Preface

Welcome to *Title*, an OpenStax resource. This textbook was written to increase student access to high-quality learning materials, maintaining highest standards of academic rigor at little to no cost.

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About OpenStax Resources

Customization

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Format

You can access this textbook for free in web view or PDF through openstax.org, and for a low cost in print.

About *Title*

Title is designed to meet the scope and sequence requirements of [include typical course title, and semester length of course if pertinent]. [List notable elements of the book that are important for this subject; specify the targeted student audience if not for a general audience. Blurb to be approved by OSX marketing team.]

Coverage and Scope

Title follows a nontraditional approach in its presentation of content. Building on the content in *Prealgebra*, the material is presented as a sequence of small steps so that students gain confidence in their ability to succeed in the course. The order of topics was carefully planned to emphasize the logical progression through the course and to facilitate a thorough understanding of each concept. As new ideas are presented, they are explicitly related to previous topics.

- **Chapter 1: Foundations**
  Chapter 1 reviews arithmetic operations with whole numbers, integers, fractions, and decimals, to give the student a solid base that will support their study of algebra.
- **Chapter 2: Solving Linear Equations and Inequalities**
  In Chapter 2, students learn to verify a solution of an equation, solve equations using the Subtraction and Addition Properties of Equality, solve equations using the Multiplication and Division Properties of Equality, solve equations with variables and constants on both sides, use a general strategy to solve linear equations, solve equations with fractions or decimals, solve a formula for a specific variable, and solve linear inequalities.
- **Chapter 3: Math Models**
  Once students have learned the skills needed to solve equations, they apply these skills in Chapter 3 to solve word and number problems.
- **Chapter 4: Graphs**
  Chapter 4 covers the rectangular coordinate system, which is the basis for most consumer graphs. Students learn to plot points on a rectangular coordinate system, graph linear equations in two variables, graph with intercepts, understand slope of a line, use the slope-intercept form of an equation of a line, find the equation of a line, and create graphs of linear inequalities.

Key Features and Boxes

Examples Each learning objective is supported by one or more worked examples, which demonstrate the problem-solving approaches that students must master. Typically, we include multiple examples for each learning

objective to model different approaches to the same type of problem, or to introduce similar problems of increasing complexity.

All examples follow a simple two- or three-part format. First, we pose a problem or question. Next, we demonstrate the solution, spelling out the steps along the way. Finally (for select examples), we show students how to check the solution. Most examples are written in a two-column format, with explanation on the left and math on the right to mimic the way that instructors “talk through” examples as they write on the board in class.

Test H4

Test

Additional Resources

Student and Instructor Resources

We’ve compiled additional resources for both students and instructors, including Getting Started Guides, [other resources dependent on book]. Instructor resources require a verified instructor account, which can be requested on your openstax.org log-in. Take advantage of these resources to supplement your OpenStax book.

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Demand, Supply, and Efficiency
By the end of this section, you will be able to:

- Contrast consumer surplus, producer surplus, and social surplus
- Explain why price floors and price ceilings can be inefficient
- Analyze demand and supply as a social adjustment mechanism

The familiar demand and supply diagram holds within it the concept of economic efficiency. One typical way that economists define efficiency is when it is impossible to improve the situation of one party without imposing a cost on another. Conversely, if a situation is inefficient, it becomes possible to benefit at least one party without imposing costs on others.

Efficiency in the demand and supply model has the same basic meaning: The economy is getting as much benefit as possible from its scarce resources and all the possible gains from trade have been achieved. In other words, the optimal amount of each good and service is being produced and consumed.

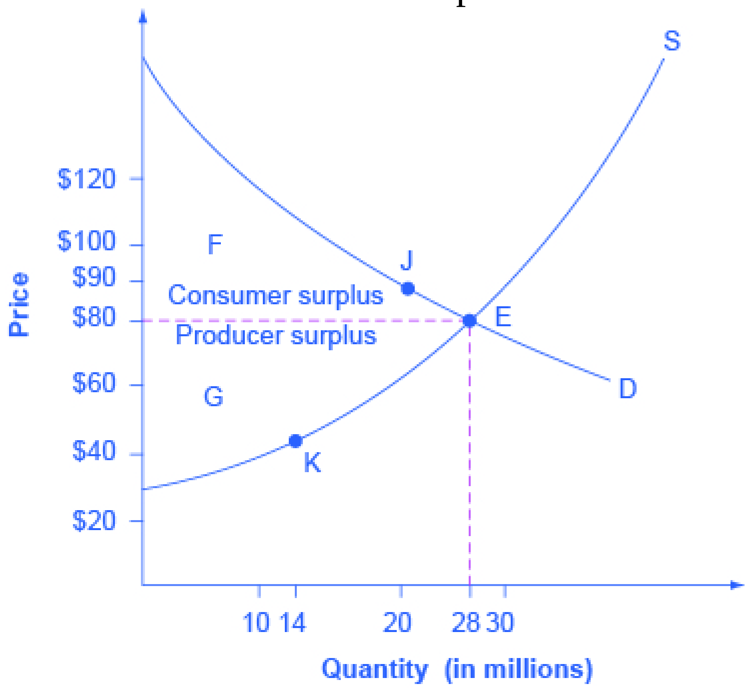
Consumer Surplus, Producer Surplus, Social Surplus

Consider a market for tablet computers, as shown in [link]. The equilibrium price is $80 and the equilibrium quantity is 28 million. To see the benefits to consumers, look at the segment of the demand curve above the equilibrium point and to the left. This portion of the demand curve shows that at least some demanders would have been willing to pay more than $80 for a tablet.

For example, point J shows that if the price was $90, 20 million tablets would be sold. Those consumers who would have been willing to pay $90 for a tablet based on the utility they expect to receive from it, but who were able to pay the equilibrium price of $80, clearly received a benefit beyond what they had to pay for. Remember, the demand curve traces consumers' willingness to pay for different quantities. The amount that individuals would have been willing to pay, minus the amount that they actually paid, is

called **consumer surplus**. Consumer surplus is the area labeled F—that is, the area above the market price and below the demand curve.

Consumer and Producer Surplus



The somewhat triangular area labeled by F shows the area of consumer surplus, which shows that the equilibrium price in the market was less than what many of the consumers were willing to pay. Point J on the demand curve shows that, even at the price of $90, consumers would have been willing to purchase a quantity of 20 million. The somewhat triangular area labeled by G shows the area of producer surplus, which shows that the equilibrium price received in the market was more than what many of the producers were willing to accept for their products. For example, point K on the supply curve shows that at a price of $45, firms would have been willing to supply a quantity of 14 million.

The supply curve shows the quantity that firms are willing to supply at each price. For example, point K in [link] illustrates that, at $45, firms would still have been willing to supply a quantity of 14 million. Those producers who would have been willing to supply the tablets at $45, but who were instead able to charge the equilibrium price of $80, clearly received an extra benefit beyond what they required to supply the product. The amount that a seller is paid for a good minus the seller's actual cost is called **producer surplus**. In [link], producer surplus is the area labeled G—that is, the area between the market price and the segment of the supply curve below the equilibrium.

The sum of consumer surplus and producer surplus is **social surplus**, also referred to as **economic surplus** or **total surplus**. In [link], social surplus would be shown as the area F + G. Social surplus is larger at equilibrium quantity and price than it would be at any other quantity. This demonstrates the economic efficiency of the market equilibrium. In addition, at the efficient level of output, it is impossible to produce greater consumer surplus without reducing producer surplus, and it is impossible to produce greater producer surplus without reducing consumer surplus.

Inefficiency of Price Floors and Price Ceilings

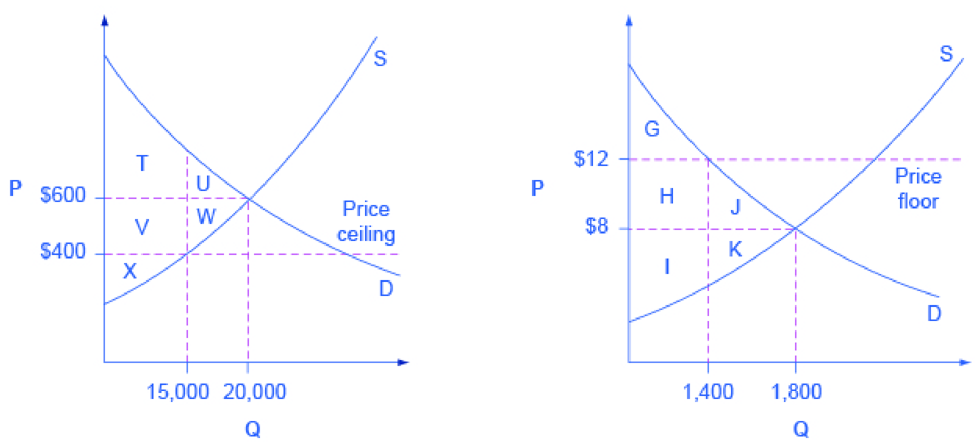
The imposition of a price floor or a price ceiling will prevent a market from adjusting to its equilibrium price and quantity, and thus will create an inefficient outcome. But there is an additional twist here. Along with creating inefficiency, price floors and ceilings will also transfer some consumer surplus to producers, or some producer surplus to consumers.

Imagine that several firms develop a promising but expensive new drug for treating back pain. If this therapy is left to the market, the equilibrium price will be $600 per month and 20,000 people will use the drug, as shown in [link] (a). The original level of consumer surplus is T + U and producer surplus is V + W + X. However, the government decides to impose a price ceiling of $400 to make the drug more affordable. At this price ceiling, firms in the market now produce only 15,000.

As a result, two changes occur. First, an inefficient outcome occurs and the total surplus of society is reduced. The loss in social surplus that occurs when the economy produces at an inefficient quantity is called **deadweight loss**. In a very real sense, it is like money thrown away that benefits no one. In [link] (a), the deadweight loss is the area U + W. When deadweight loss exists, it is possible for both consumer and producer surplus to be higher, in this case because the price control is blocking some suppliers and demanders from transactions they would both be willing to make.

A second change from the price ceiling is that some of the producer surplus is transferred to consumers. After the price ceiling is imposed, the new consumer surplus is T + V, while the new producer surplus is X. In other words, the price ceiling transfers the area of surplus (V) from producers to consumers. Note that the gain to consumers is less than the loss to producers, which is just another way of seeing the deadweight loss.

Efficiency and Price Floors and Ceilings



(a) Reduced social surplus from a price ceiling (b) Reduced social surplus from a price floor

(a) The original equilibrium price is $600 with a quantity of 20,000. Consumer surplus is T + U, and producer surplus is V + W + X. A price ceiling is imposed at $400, so firms in the market now produce only a quantity of 15,000. As a result, the new consumer surplus is T + V, while the new producer surplus is X. (b) The original equilibrium is $8 at a quantity of 1,800. Consumer surplus is G + H + J, and producer surplus is I + K. A price floor is imposed at $12, which means that quantity demanded falls to 1,400. As a result, the new consumer surplus is G, and the new producer surplus is H + I.

[link] (b) shows a price floor example using a string of struggling movie theaters, all in the same city. The current equilibrium is $8 per movie ticket, with 1,800 people attending movies. The original consumer surplus is G + H + J, and producer surplus is I + K. The city government is worried that movie theaters will go out of business, reducing the entertainment options available to citizens, so it decides to impose a price floor of $12 per ticket. As a result, the quantity demanded of movie tickets falls to 1,400. The new consumer surplus is G, and the new producer surplus is H + I. In effect, the price floor causes the area H to be transferred from consumer to producer surplus, but also causes a deadweight loss of J + K.

This analysis shows that a price ceiling, like a law establishing rent controls, will transfer some producer surplus to consumers—which helps to explain why consumers often favor them. Conversely, a price floor like a guarantee that farmers will receive a certain price for their crops will transfer some consumer surplus to producers, which explains why producers often favor them. However, both price floors and price ceilings block some transactions that buyers and sellers would have been willing to make, and creates deadweight loss. Removing such barriers, so that prices and quantities can adjust to their equilibrium level, will increase the economy's social surplus.

Demand and Supply as a Social Adjustment Mechanism

The demand and supply model emphasizes that prices are not set only by demand or only by supply, but by the interaction between the two. In 1890, the famous economist Alfred Marshall wrote that asking whether supply or demand determined a price was like arguing "whether it is the upper or the under blade of a pair of scissors that cuts a piece of paper." The answer is that both blades of the demand and supply scissors are always involved.

The adjustments of equilibrium price and quantity in a market-oriented economy often occur without much government direction or oversight. If the coffee crop in Brazil suffers a terrible frost, then the supply curve of

coffee shifts to the left and the price of coffee rises. Some people—call them the coffee addicts—continue to drink coffee and pay the higher price. Others switch to tea or soft drinks. No government commission is needed to figure out how to adjust coffee prices, which companies will be allowed to process the remaining supply, which supermarkets in which cities will get how much coffee to sell, or which consumers will ultimately be allowed to drink the brew. Such adjustments in response to price changes happen all the time in a market economy, often so smoothly and rapidly that we barely notice them.

Think for a moment of all the seasonal foods that are available and inexpensive at certain times of the year, like fresh corn in midsummer, but more expensive at other times of the year. People alter their diets and restaurants alter their menus in response to these fluctuations in prices without fuss or fanfare. For both the U.S. economy and the world economy as a whole, markets—that is, demand and supply—are the primary social mechanism for answering the basic questions about what is produced, how it is produced, and for whom it is produced.

Note:
Why Can We Not Get Enough of Organic?
Organic food is grown without synthetic pesticides, chemical fertilizers or genetically modified seeds. In recent decades, the demand for organic products has increased dramatically. The Organic Trade Association reported sales increased from $1 billion in 1990 to $35.1 billion in 2013, more than 90% of which were sales of food products.
Why, then, are organic foods more expensive than their conventional counterparts? The answer is a clear application of the theories of supply and demand. As people have learned more about the harmful effects of chemical fertilizers, growth hormones, pesticides and the like from large-scale factory farming, our tastes and preferences for safer, organic foods have increased. This change in tastes has been reinforced by increases in income, which allow people to purchase pricier products, and has made organic foods more mainstream. This has led to an increased demand for organic foods. Graphically, the demand curve has shifted right, and we

have moved up the supply curve as producers have responded to the higher prices by supplying a greater quantity.

In addition to the movement along the supply curve, we have also had an increase in the number of farmers converting to organic farming over time. This is represented by a shift to the right of the supply curve. Since both demand and supply have shifted to the right, the resulting equilibrium quantity of organic foods is definitely higher, but the price will only fall when the increase in supply is larger than the increase in demand. We may need more time before we see lower prices in organic foods. Since the production costs of these foods may remain higher than conventional farming, because organic fertilizers and pest management techniques are more expensive, they may never fully catch up with the lower prices of non-organic foods.

As a final, specific example: The Environmental Working Group's "Dirty Dozen" list of fruits and vegetables, which test high for pesticide residue even after washing, was released in April 2013. The inclusion of strawberries on the list has led to an increase in demand for organic strawberries, resulting in both a higher equilibrium price and quantity of sales.

Consumer surplus is the gap between the price that consumers are willing to pay, based on their preferences, and the market equilibrium price. Producer surplus is the gap between the price for which producers are willing to sell a product, based on their costs, and the market equilibrium price. Social surplus is the sum of consumer surplus and producer surplus. Total surplus is larger at the equilibrium quantity and price than it will be at any other quantity and price. Deadweight loss is loss in total surplus that occurs when the economy produces at an inefficient quantity.

Exercise:

Problem:

Does a price ceiling increase or decrease the number of transactions in a market? Why? What about a price floor?

Solution:

Assuming that people obey the price ceiling, the market price will be below equilibrium, which means that Qd will be more than Qs. Buyers can only buy what is offered for sale, so the number of transactions will fall to Qs. This is easy to see graphically. By analogous reasoning, with a price floor the market price will be above the equilibrium price, so Qd will be less than Qs. Since the limit on transactions here is demand, the number of transactions will fall to Qd. Note that because both price floors and price ceilings reduce the number of transactions, social surplus is less.

Exercise:

Problem:

If a price floor benefits producers, why does a price floor reduce social surplus?

Solution:

Because the losses to consumers are greater than the benefits to producers, so the net effect is negative. Since the lost consumer surplus is greater than the additional producer surplus, social surplus falls.

Exercise:

Problem:

What is consumer surplus? How is it illustrated on a demand and supply diagram?

Exercise:

Problem:

What is producer surplus? How is it illustrated on a demand and supply diagram?

Exercise:

Problem:

What is total surplus? How is it illustrated on a demand and supply diagram?

Exercise:

Problem:

What is the relationship between total surplus and economic efficiency?

Exercise:

Problem: What is deadweight loss?

Exercise:

Problem:

What term would an economist use to describe what happens when a shopper gets a “good deal” on a product?

Exercise:

Problem: Explain why voluntary transactions improve social welfare.

Exercise:

Problem:

Why would a free market never operate at a quantity greater than the equilibrium quantity? *Hint:* What would be required for a transaction to occur at that quantity?

Glossary

consumer surplus
: the extra benefit consumers receive from buying a good or service, measured by what the individuals would have been willing to pay

minus the amount that they actually paid

deadweight loss
: the loss in social surplus that occurs when a market produces an inefficient quantity

economic surplus
: see social surplus

producer surplus
: the extra benefit producers receive from selling a good or service, measured by the price the producer actually received minus the price the producer would have been willing to accept

social surplus
: the sum of consumer surplus and producer surplus

Sex and Gender

- Define and differentiate between sex and gender
- Define and discuss what is meant by gender identity
- Understand and discuss the role of homophobia and heterosexism in society
- Distinguish the meanings of transgender, transsexual, and homosexual identities

While the biological differences between males and females are fairly straightforward, the social and cultural aspects of being a man or woman can be complicated. (Photo courtesy of FaceMePLS/flickr)

When filling out a document such as a job application or school registration form you are often asked to provide your name, address, phone number, birth date, and sex or gender. But have you ever been asked to provide your sex *and* your gender? Like most people, you may not have realized that sex and gender are not the same. However, sociologists and most other social scientists view them as conceptually distinct. **Sex** refers to physical or physiological differences between males and females, including both primary sex characteristics (the reproductive system) and secondary

characteristics such as height and muscularity. **Gender** refers to behaviors, personal traits, and social positions that society attributes to being female or male.

A person’s sex, as determined by his or her biology, does not always correspond with his or her gender. Therefore, the terms *sex* and *gender* are not interchangeable. A baby boy who is born with male genitalia will be identified as male. As he grows, however, he may identify with the feminine aspects of his culture. Since the term *sex* refers to biological or physical distinctions, characteristics of sex will not vary significantly between different human societies. Generally, persons of the female sex, regardless of culture, will eventually menstruate and develop breasts that can lactate. Characteristics of gender, on the other hand, may vary greatly between different societies. For example, in U.S. culture, it is considered feminine (or a trait of the female gender) to wear a dress or skirt. However, in many Middle Eastern, Asian, and African cultures, dresses or skirts (often referred to as sarongs, robes, or gowns) are considered masculine. The kilt worn by a Scottish male does not make him appear feminine in his culture.

The dichotomous view of gender (the notion that someone is either male or female) is specific to certain cultures and is not universal. In some cultures gender is viewed as fluid. In the past, some anthropologists used the term *berdache* to refer to individuals who occasionally or permanently dressed and lived as a different gender. The practice has been noted among certain Native American tribes (Jacobs, Thomas, and Lang 1997). Samoan culture accepts what Samoans refer to as a “third gender.” *Fa’afafine*, which translates as “the way of the woman,” is a term used to describe individuals who are born biologically male but embody both masculine and feminine traits. Fa’afafines are considered an important part of Samoan culture. Individuals from other cultures may mislabel them as homosexuals because fa’afafines have a varied sexual life that may include men and women (Poasa 1992).

Note:
The Legalese of Sex and Gender

The terms *sex* and *gender* have not always been differentiated in the English language. It was not until the 1950s that U.S. and British psychologists and other professionals working with intersex and transsexual patients formally began distinguishing between sex and gender. Since then, psychological and physiological professionals have increasingly used the term gender (Moi 2005). By the end of the twenty-first century, expanding the proper usage of the term *gender* to everyday language became more challenging—particularly where legal language is concerned. In an effort to clarify usage of the terms *sex* and *gender*, U.S. Supreme Court Justice Antonin Scalia wrote in a 1994 briefing, "The word gender has acquired the new and useful connotation of cultural or attitudinal characteristics (as opposed to physical characteristics) distinctive to the sexes. That is to say, gender is to sex as feminine is to female and masculine is to male" (*J.E.B. v. Alabama*, 144 S. Ct. 1436 [1994]). Supreme Court Justice Ruth Bader Ginsburg had a different take, however. Viewing the words as synonymous, she freely swapped them in her briefings so as to avoid having the word "sex" pop up too often. It is thought that her secretary supported this practice by suggestions to Ginsberg that "those nine men" (the other Supreme Court justices), "hear that word and their first association is not the way you want them to be thinking" (Case 1995). This anecdote reveals that both sex and gender are actually socially defined variables whose definitions change over time.

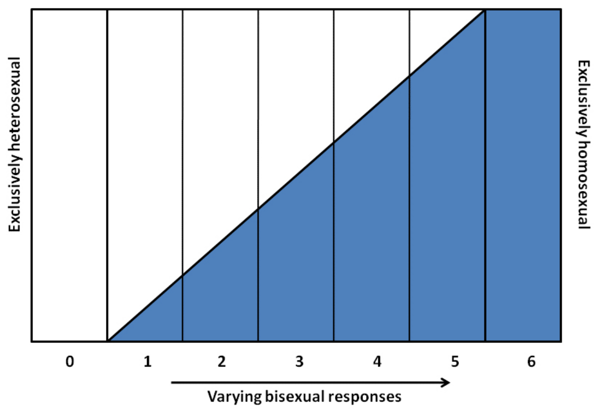
Sexual Orientation

A person's **sexual orientation** is his or her physical, mental, emotional, and sexual attraction to a particular sex (male or female). Sexual orientation is typically divided into four categories: *heterosexuality*, the attraction to individuals of the other sex; *homosexuality*, the attraction to individuals of the same sex; *bisexuality*, the attraction to individuals of either sex; and *asexuality*, no attraction to either sex. Heterosexuals and homosexuals may also be referred to informally as "straight" and "gay," respectively. The United States is a **heteronormative society**, meaning it assumes sexual orientation is biologically determined and unambiguous. Consider that homosexuals are often asked, "When did you know you were gay?" but

heterosexuals are rarely asked, “When did you know that you were straight?” (Ryle 2011).

According to current scientific understanding, individuals are usually aware of their sexual orientation between middle childhood and early adolescence (American Psychological Association 2008). They do not have to participate in sexual activity to be aware of these emotional, romantic, and physical attractions; people can be celibate and still recognize their sexual orientation. Homosexual women (also referred to as lesbians), homosexual men (also referred to as gays), and bisexuals of both genders may have very different experiences of discovering and accepting their sexual orientation. At the point of puberty, some may be able to announce their sexual orientations, while others may be unready or unwilling to make their homosexuality or bisexuality known since it goes against U.S. society’s historical norms (APA 2008).

Alfred Kinsey was among the first to conceptualize sexuality as a continuum rather than a strict dichotomy of gay or straight. He created a six-point rating scale that ranges from exclusively heterosexual to exclusively homosexual. See the figure below. In his 1948 work *Sexual Behavior in the Human Male*, Kinsey writes, “Males do not represent two discrete populations, heterosexual and homosexual. The world is not to be divided into sheep and goats … The living world is a continuum in each and every one of its aspects” (Kinsey 1948).



The Kinsey scale indicates that

sexuality can be measured by
more than just heterosexuality
and homosexuality.

Later scholarship by Eve Kosofsky Sedgwick expanded on Kinsey’s notions. She coined the term “homosocial” to oppose “homosexual,” describing nonsexual same-sex relations. Sedgwick recognized that in U.S. culture, males are subject to a clear divide between the two sides of this continuum, whereas females enjoy more fluidity. This can be illustrated by the way women in the United States can express homosocial feelings (nonsexual regard for people of the same sex) through hugging, handholding, and physical closeness. In contrast, U.S. males refrain from these expressions since they violate the heteronormative expectation that male sexual attraction should be exclusively for females. Research suggests that it is easier for women violate these norms than men, because men are subject to more social disapproval for being physically close to other men (Sedgwick 1985).

There is no scientific consensus regarding the exact reasons why an individual holds a heterosexual, homosexual, or bisexual orientation. Research has been conducted to study the possible genetic, hormonal, developmental, social, and cultural influences on sexual orientation, but there has been no evidence that links sexual orientation to one factor (APA 2008). Research, however, does present evidence showing that homosexuals and bisexuals are treated differently than heterosexuals in schools, the workplace, and the military. In 2011, for example, Sears and Mallory used General Social Survey data from 2008 to show that 27 percent of lesbian, gay, bisexual (LGB) respondents reported experiencing sexual orientation-based discrimination during the five years prior to the survey. Further, 38 percent of openly LGB people experienced discrimination during the same time.

Much of this discrimination is based on stereotypes and misinformation. Some is based on **heterosexism**, which Herek (1990) suggests is both an ideology and a set of institutional practices that privilege heterosexuals and

heterosexuality over other sexual orientations. Much like racism and sexism, heterosexism is a systematic disadvantage embedded in our social institutions, offering power to those who conform to hetereosexual orientation while simultaneously disadvantaging those who do not. *Homophobia*, an extreme or irrational aversion to homosexuals, accounts for further stereotyping and discrimination. Major policies to prevent discrimination based on sexual orientation have not come into effect until the last few years. In 2011, President Obama overturned “don’t ask, don’t tell,” a controversial policy that required homosexuals in the US military to keep their sexuality undisclosed. The Employee Non-Discrimination Act, which ensures workplace equality regardless of sexual orientation, is still pending full government approval. Organizations such as GLAAD (Gay & Lesbian Alliance Against Defamation) advocate for homosexual rights and encourage governments and citizens to recognize the presence of sexual discrimination and work to prevent it. Other advocacy agencies frequently use the acronyms LBGT and LBGTQ, which stands for “Lesbian, Gay, Bisexual, Transgender” (and “Queer” or “Questioning” when the Q is added).

Sociologically, it is clear that gay and lesbian couples are negatively affected in states where they are denied the legal right to marriage. In 1996, The Defense of Marriage Act (**DOMA**) was passed, explicitly limiting the definition of “marriage” to a union between one man and one woman. It also allowed individual states to choose whether or not they recognized same-sex marriages performed in other states. Imagine that you married an opposite-sex partner under similar conditions—if you went on a cross-country vacation the validity of your marriage would change every time you crossed state lines. In another blow to same-sex marriage advocates, in November 2008 California passed Proposition 8, a state law that limited marriage to unions of opposite-sex partners.

Over time, advocates for same-sex marriage have won several court cases, laying the groundwork for legalized same-sex marriage across the United States, including the June 2013 decision to overturn part of DOMA in *Windsor v. United States*, and the Supreme Court’s dismissal of *Hollingsworth v. Perry*, affirming the August 2010 ruling that found California’s Proposition 8 unconstitutional. In October 2014, the U.S.

Supreme Court declined to hear appeals to rulings against same-sex marriage bans, which effectively legalized same-sex marriage in Indiana, Oklahoma, Utah, Virginia, and Wisconsin, Colorado, North Carolina, West Virginia, and Wyoming (Freedom to Marry, Inc. 2014). Same-sex marriage is now legal across most of the United States. The next few years will determine whether the right to same-sex marriage is affirmed, depending on whether the U.S. Supreme Court takes a judicial step to guarantee the freedom to marry as a civil right.

Gender Roles

As we grow, we learn how to behave from those around us. In this socialization process, children are introduced to certain roles that are typically linked to their biological sex. The term **gender role** refers to society's concept of how men and women are expected to look and how they should behave. These roles are based on norms, or standards, created by society. In U.S. culture, masculine roles are usually associated with strength, aggression, and dominance, while feminine roles are usually associated with passivity, nurturing, and subordination. Role learning starts with socialization at birth. Even today, our society is quick to outfit male infants in blue and girls in pink, even applying these color-coded gender labels while a baby is in the womb.

One way children learn gender roles is through play. Parents typically supply boys with trucks, toy guns, and superhero paraphernalia, which are active toys that promote motor skills, aggression, and solitary play. Daughters are often given dolls and dress-up apparel that foster nurturing, social proximity, and role play. Studies have shown that children will most likely choose to play with "gender appropriate" toys (or same-gender toys) even when cross-gender toys are available because parents give children positive feedback (in the form of praise, involvement, and physical closeness) for gender normative behavior (Caldera, Huston, and O'Brien 1998).

Fathers tend to be more involved when their sons engage in gender-appropriate activities such as sports. (Photo courtesy of Shawn Lea/flickr)

The drive to adhere to masculine and feminine gender roles continues later in life. Men tend to outnumber women in professions such as law enforcement, the military, and politics. Women tend to outnumber men in care-related occupations such as childcare, healthcare (even though the term “doctor” still conjures the image of a man), and social work. These occupational roles are examples of typical U.S. male and female behavior, derived from our culture’s traditions. Adherence to them demonstrates fulfillment of social expectations but not necessarily personal preference (Diamond 2002).

Gender Identity

U.S. society allows for some level of flexibility when it comes to acting out gender roles. To a certain extent, men can assume some feminine roles and

women can assume some masculine roles without interfering with their gender identity. **Gender identity** is a person's deeply held internal perception of his or her gender.

Individuals who identify with the role that is the different from their biological sex are called **transgender**. Transgender is not the same as homosexual, and many homosexual males view both their sex and gender as male. Transgender females are males who have such a strong emotional and psychological connection to the feminine aspects of society that they identify their gender as female. The parallel connection to masculinity exists for transgender males. It is difficult to determine the prevalence of transgenderism in society. However, it is estimated that two to five percent of the U.S. population is transgender (Transgender Law and Policy Institute 2007).

Transgender individuals who attempt to alter their bodies through medical interventions such as surgery and hormonal therapy—so that their physical being is better aligned with gender identity—are called **transsexuals**. They may also be known as male-to-female (MTF) or female-to-male (FTM). Not all transgender individuals choose to alter their bodies: many will maintain their original anatomy but may present themselves to society as another gender. This is typically done by adopting the dress, hairstyle, mannerisms, or other characteristic typically assigned to another gender. It is important to note that people who cross-dress, or wear clothing that is traditionally assigned to a gender different from their biological sex, are not necessarily transgender. Cross-dressing is typically a form of self-expression, entertainment, or personal style, and it is not necessarily an expression against one's assigned gender (APA 2008).

There is no single, conclusive explanation for why people are transgender. Transgender expressions and experiences are so diverse that it is difficult to identify their origin. Some hypotheses suggest biological factors such as genetics or prenatal hormone levels as well as social and cultural factors such as childhood and adulthood experiences. Most experts believe that all of these factors contribute to a person's gender identity (APA 2008).

After years of controversy over the treatment of sex and gender in the *American Psychiatric Association Diagnostic and Statistical Manual for*

Mental Disorders (Drescher 2010), the most recent edition, DSM-5, responds to allegations that the term “Gender Identity Disorder” is stigmatizing by replacing it with “Gender Dysphoria.” Gender Identity Disorder as a diagnostic category stigmatized the patient by implying there was something “disordered” about them. Gender Dysphoria, on the other hand, removes some of that stigma by taking the word "disorder" out while maintaining a category that will protect patient access to care, including hormone therapy and gender reassignment surgery. In the DSM-5, **Gender Dysphoria** is a condition of people whose gender at birth is contrary to the one they identify with. For a person to be diagnosed with Gender Dysphoria, there must be a marked difference between the individual’s expressed/experienced gender and the gender others would assign him or her, and it must continue for at least six months. In children, the desire to be of the other gender must be present and verbalized. This diagnosis is now a separate category from sexual dysfunction and paraphilia, another important part of removing stigma from the diagnosis (APA 2013).

Changing the clinical description may contribute to a larger acceptance of transgender people in society. Studies show that people who identify as transgender are twice as likely to experience assault or discrimination as nontransgender individuals; they are also one and a half times more likely to experience intimidation (National Coalition of Anti-Violence Programs 2010; Giovanniello 2013). Organizations such as the National Coalition of Anti-Violence Programs and Global Action for Trans Equality work to prevent, respond to, and end all types of violence against transgender, transsexual, and homosexual individuals. These organizations hope that by educating the public about gender identity and empowering transgender and transsexual individuals, this violence will end.

Note:

Real-Life *Freaky Friday*

What if you had to live as a sex you were not biologically born to? If you are a man, imagine that you were forced to wear frilly dresses, dainty shoes, and makeup to special occasions, and you were expected to enjoy romantic comedies and daytime talk shows. If you are a woman, imagine

that you were forced to wear shapeless clothing, put only minimal effort into your personal appearance, not show emotion, and watch countless hours of sporting events and sports-related commentary. It would be pretty uncomfortable, right? Well, maybe not. Many people enjoy participating in activities, whether they are associated with their biological sex or not, and would not mind if some of the cultural expectations for men and women were loosened.

Now, imagine that when you look at your body in the mirror, you feel disconnected. You feel your genitals are shameful and dirty, and you feel as though you are trapped in someone else's body with no chance of escape. As you get older, you hate the way your body is changing, and, therefore, you hate yourself. These elements of disconnect and shame are important to understand when discussing transgender individuals. Fortunately, sociological studies pave the way for a deeper and more empirically grounded understanding of the transgender experience.

Chaz Bono is the transgender son of Cher and Sonny Bono. While he was born female, he considers

himself male. Being transgender is not about clothing or hairstyles; it is about self-perception. (Photo courtesy of Greg Hernandez/flickr)

Summary

The terms “sex” and “gender” refer to two different identifiers. Sex denotes biological characteristics differentiating males and females, while gender denotes social and cultural characteristics of masculine and feminine behavior. Sex and gender are not always synchronous. Individuals who strongly identify with the opposing gender are considered transgender.

Section Quiz

Exercise:

Problem:

The terms “masculine” and “feminine” refer to a person’s __________.

a. sex
b. gender
c. both sex and gender
d. none of the above

Solution:
Answer

B

Exercise:

Problem:

The term ________ refers to society's concept of how men and women are expected to act and how they should behave.

a. gender role
b. gender bias
c. sexual orientation
d. sexual attitudes

Solution:
Answer

A

Exercise:

Problem:

Research indicates that individuals are aware of their sexual orientation ________.

a. at infancy
b. in early adolescence
c. in early adulthood
d. in late adulthood

Solution:
Answer

B

Exercise:

Problem:

A person who is biologically female but identifies with the male gender and has undergone surgery to alter her body is considered _______.

a. transgender
b. transsexual
c. a cross-dresser
d. homosexual

Solution:
Answer

B

Exercise:

Problem:

Which of following is correct regarding the explanation for transgenderism?

a. It is strictly biological and associated with chemical imbalances in the brain.
b. It is a behavior that is learned through socializing with other transgender individuals.
c. It is genetic and usually skips one generation.
d. Currently, there is no definitive explanation for transgenderism.

Solution:
Answer

D

Short Answer

Exercise:

Problem:

Why do sociologists find it important to differentiate between sex and gender? What importance does the differentiation have in modern society?

Exercise:

Problem:

How is children's play influenced by gender roles? Think back to your childhood. How "gendered" were the toys and activities available to you? Do you remember gender expectations being conveyed through the approval or disapproval of your playtime choices?

Further Research

For more information on gender identity and advocacy for transgender individuals see the Global Action for Trans Equality web site at http://openstaxcollege.org/l/trans_equality.

References

American Psychological Association (APA). 2008. "Answers to Your Questions: For a Better Understanding of Sexual Orientation and Homosexuality." Washington, DC. Retrieved January 10, 2012 (http://www.apa.org/topics/sexuality/orientation.aspx).

American Psychiatric Association. 2013. *Diagnostic and Statistical Manual of Mental Disorders* (5th ed.). Washington, DC: American Psychiatric Association.

Caldera, Yvonne, Aletha Huston, and Marion O'Brien. 1998. "Social Interactions and Play Patterns of Parents and Toddlers with Feminine, Masculine, and Neutral Toys." *Child Development* 60(1):70–76.

Case, M.A. 1995. "Disaggregating Gender from Sex and Sexual Orientation: The Effeminate Man in the Law and Feminist Jurisprudence." *Yale Law Journal* 105(1):1–105.

Drescher, J. 2010. “Queer diagnoses: Parallels and contrasts in the history of homosexuality, gender variance, and the Diagnostic and Statistical Manual. Archives of Sexual Behavior.” 39: 427–460.

Freedom to Marry, Inc. 2014. "History and Timeline of the Freedom to Marry in the United States | Freedom to Marry.” Retrieved November 11, 2014 (http://www.freedomtomarry.org/pages/history-and-timeline-of-marriage).

Giovanniello, Sarah. 2013. "NCAVP Report: 2012 Hate Violence Disproportionately Target Transgender Women of Color." GLAAD. N.p., Retrieved October 10, 2014 (http://www.glaad.org/blog/ncavp-report-2012-hate-violence-disproportionately-target-transgender-women-color).

Herek, G. M. 1990. “The Context of Anti-Gay Violence: Notes on Cultural and Psychological Heterosexism." *Journal of Interpersonal Violence*. 5: 316–333.

Jacobs, Sue-Ellen, Wesley Thomas, and Sabine Lang. 1997. *Two Spirit People: Native American Gender Identity, Sexuality, and Spirituality*. Champaign, IL: University of Illinois Press.

J.E.B. v. Alabama, 144 S. Ct. 1436 (1994).

Kinsey, Alfred C. et al. 1998 [1948]. *Sexual Behavior in the Human Male*. Bloomington, IN: Indiana University Press.

Moi, T. 2005. *Sex, Gender and the Body*. New York: Oxford University Press.

National Coalition of Anti-Violence Programs. 2010. “Hate Violence Against Lesbian, Gay, Bisexual, Transgender, Queer and HIV-Affected Communities in the United States.” Retrieved January 10, 2012

(http://www.avp.org/storage/documents/Reports/2012_NCAVP_2011_HV_Report.pdf).

Poasa, Kris. 1992. “The Samoan Fa’afafine: One Case Study and Discussion of Transsexualism.” *Journal of Psychology & Human Sexuality* 5(3):39–51.

Ryle, Robyn. 2011. *Questioning Gender: A Sociological Exploration.* Thousand Oaks, CA: Pine Forge Press.

Sears, Brad and Christy Mallory. 2011. "Documented Evidence of Employment Discrimination & Its Effects on LGBT People." Los Angeles, CA: The Williams Institute. Retrieved December 12, 2014 (http://williamsinstitute.law.ucla.edu/wp-content/uploads/Sears-Mallory-Discrimination-July-20111.pdf)

Sedgwick, Eve. 1985. *Between Men: English Literature and Male Homosocial Desire*. New York: Columbia University Press.

Glossary

DOMA
: Defense of Marriage Act, a 1996 U.S. law explicitly limiting the definition of “marriage” to a union between one man and one woman and allowing each individual state to recognize or deny same-sex marriages performed in other states

gender dysphoria
: a condition listed in the DSM-5 in which people whose gender at birth is contrary to the one they identify with. This condition replaces "gender identity disorder"

gender identity
: a person’s deeply held internal perception of his or her gender

gender role
: society’s concept of how men and women should behave

gender

a term that refers to social or cultural distinctions of behaviors that are considered male or female

heterosexism

an ideology and a set of institutional practices that privilege heterosexuals and heterosexuality over other sexual orientations

homophobia

an extreme or irrational aversion to homosexuals

sex

a term that denotes the presence of physical or physiological differences between males and females

sexual orientation

a person’s physical, mental, emotional, and sexual attraction to a particular sex (male or female)

transgender

an adjective that describes individuals who identify with the behaviors and characteristics that are other than their biological sex

transsexuals

transgender individuals who attempt to alter their bodies through medical interventions such as surgery and hormonal therapy

Exponents and Scientific Notation
In this section students will:

- Use the product rule of exponents.
- Use the quotient rule of exponents.
- Use the power rule of exponents.
- Use the zero exponent rule of exponents.
- Use the negative rule of exponents.
- Find the power of a product and a quotient.
- Simplify exponential expressions.
- Use scientific notation.

Mathematicians, scientists, and economists commonly encounter very large and very small numbers. But it may not be obvious how common such figures are in everyday life. For instance, a pixel is the smallest unit of light that can be perceived and recorded by a digital camera. A particular camera might record an image that is 2,048 pixels by 1,536 pixels, which is a very high resolution picture. It can also perceive a color depth (gradations in colors) of up to 48 bits per frame, and can shoot the equivalent of 24 frames per second. The maximum possible number of bits of information used to film a one-hour (3,600-second) digital film is then an extremely large number.

Using a calculator, we enter $2{,}048 \times 1{,}536 \times 48 \times 24 \times 3{,}600$ and press ENTER. The calculator displays 1.304596316E13. What does this mean? The “E13” portion of the result represents the exponent 13 of ten, so there are a maximum of approximately 1.3×10^{13} bits of data in that one-hour film. In this section, we review rules of exponents first and then apply them to calculations involving very large or small numbers.

Using the Product Rule of Exponents

Consider the product $x^3 \cdot x^4$. Both terms have the same base, *x*, but they are raised to different exponents. Expand each expression, and then rewrite the resulting expression.
Equation:

$$\begin{aligned} x^3 \cdot x^4 &= \overbrace{x \cdot x \cdot x}^{\text{3 factors}} \cdot \overbrace{x \cdot x \cdot x \cdot x}^{\text{4 factors}} \\ &= \overbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}^{\text{7 factors}} \\ &= x^7 \end{aligned}$$

The result is that $x^3 \cdot x^4 = x^{3+4} = x^7$.

Notice that the exponent of the product is the sum of the exponents of the terms. In other words, when multiplying exponential expressions with the same base, we write the result with the common base and add the exponents. This is the *product rule of exponents.*
Equation:

$$a^m \cdot a^n = a^{m+n}$$

Now consider an example with real numbers.
Equation:

$$2^3 \cdot 2^4 = 2^{3+4} = 2^7$$

We can always check that this is true by simplifying each exponential expression. We find that 2^3 is 8, 2^4 is 16, and 2^7 is 128. The product $8 \cdot 16$ equals 128, so the relationship is true. We can use the product rule of exponents to simplify expressions that are a product of two numbers or expressions with the same base but different exponents.

Note:
The Product Rule of Exponents
For any real number a and natural numbers m and n,the product rule of exponents states that
Equation:

$$a^m \cdot a^n = a^{m+n}$$

Example:
Exercise:

Problem:
Using the Product Rule

Write each of the following products with a single base. Do not simplify further.

a. $t^5 \cdot t^3$
b. $(-3)^5 \cdot (-3)$
c. $x^2 \cdot x^5 \cdot x^3$

Solution:

Use the product rule to simplify each expression.

a. $t^5 \cdot t^3 = t^{5+3} = t^8$
b. $(-3)^5 \cdot (-3) = (-3)^5 \cdot (-3)^1 = (-3)^{5+1} = (-3)^6$
c. $x^2 \cdot x^5 \cdot x^3$

At first, it may appear that we cannot simplify a product of three factors. However, using the associative property of multiplication, begin by simplifying the first two.
Equation:

$$x^2 \cdot x^5 \cdot x^3 = \left(x^2 \cdot x^5\right) \cdot x^3 = \left(x^{2+5}\right) \cdot x^3 = x^7 \cdot x^3 = x^{7+3} = x^{10}$$

Notice we get the same result by adding the three exponents in one step.
Equation:

$$x^2 \cdot x^5 \cdot x^3 = x^{2+5+3} = x^{10}$$

Note:
Exercise:

Problem: Write each of the following products with a single base. Do not simplify further.

a. $k^6 \cdot k^9$
b. $\left(\frac{2}{y}\right)^4 \cdot \left(\frac{2}{y}\right)$
c. $t^3 \cdot t^6 \cdot t^5$

Solution:

a. k^{15}
b. $\left(\frac{2}{y}\right)^5$
c. t^{14}

Using the Quotient Rule of Exponents

The *quotient rule of exponents* allows us to simplify an expression that divides two numbers with the same base but different exponents. In a similar way to the product rule, we can simplify an expression such as $\frac{y^m}{y^n}$,where $m > n.$ Consider the example $\frac{y^9}{y^5}.$ Perform the division by canceling common factors.

Equation:

$$\begin{aligned} \frac{y^9}{y^5} &= \frac{y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y}{y \cdot y \cdot y \cdot y \cdot y} \\ &= \frac{\cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot y \cdot y \cdot y \cdot y}{\cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y}} \\ &= \frac{y \cdot y \cdot y \cdot y}{1} \\ &= y^4 \end{aligned}$$

Notice that the exponent of the quotient is the difference between the exponents of the divisor and dividend.

Equation:

$$\frac{a^m}{a^n} = a^{m-n}$$

In other words, when dividing exponential expressions with the same base, we write the result with the common base and subtract the exponents.

Equation:

$$\frac{y^9}{y^5} = y^{9-5} = y^4$$

For the time being, we must be aware of the condition $m > n.$ Otherwise, the difference $m - n$ could be zero or negative. Those possibilities will be explored shortly. Also, instead of qualifying variables as nonzero each time, we will simplify matters and assume from here on that all variables represent nonzero real numbers.

Note:
The Quotient Rule of Exponents
For any real number a and natural numbers m and n,such that $m > n$,the quotient rule of exponents states that

Equation:

$$\frac{a^m}{a^n} = a^{m-n}$$

Example:
Exercise:

Problem:
Using the Quotient Rule

Write each of the following products with a single base. Do not simplify further.

a. $\frac{(-2)^{14}}{(-2)^9}$
b. $\frac{t^{23}}{t^{15}}$
c. $\frac{\left(z\sqrt{2}\right)^5}{z\sqrt{2}}$

Solution:

Use the quotient rule to simplify each expression.

a. $\frac{(-2)^{14}}{(-2)^9} = (-2)^{14-9} = (-2)^5$
b. $\frac{t^{23}}{t^{15}} = t^{23-15} = t^8$
c. $\frac{\left(z\sqrt{2}\right)^5}{z\sqrt{2}} = \left(z\sqrt{2}\right)^{5-1} = \left(z\sqrt{2}\right)^4$

Note:
Exercise:

Problem: Write each of the following products with a single base. Do not simplify further.

a. $\frac{s^{75}}{s^{68}}$
b. $\frac{(-3)^6}{-3}$
c. $\frac{\left(ef^2\right)^5}{\left(ef^2\right)^3}$

Solution:

a. s^7
b. $(-3)^5$
c. $\left(ef^2\right)^2$

Using the Power Rule of Exponents

Suppose an exponential expression is raised to some power. Can we simplify the result? Yes. To do this, we use the *power rule of exponents*. Consider the expression $\left(x^2\right)^3$. The expression inside the parentheses is multiplied twice because it has an exponent of 2. Then the result is multiplied three times because the entire expression has an exponent of 3.

Equation:

$$\begin{aligned}\left(x^2\right)^3 &= \overbrace{(x^2)\cdot(x^2)\cdot(x^2)}^{\text{3 factors}}\\ &= \overbrace{\left(\overbrace{x\cdot x}^{\text{2 factors}}\right)\cdot\left(\overbrace{x\cdot x}^{\text{2 factors}}\right)\cdot\left(\overbrace{x\cdot x}^{\text{2 factors}}\right)}^{\text{3 factors}}\\ &= x\cdot x\cdot x\cdot x\cdot x\cdot x\\ &= x^6\end{aligned}$$

The exponent of the answer is the product of the exponents: $\left(x^2\right)^3 = x^{2\cdot 3} = x^6$. In other words, when raising an exponential expression to a power, we write the result with the common base and the product of the exponents.

Equation:

$$(a^m)^n = a^{m\cdot n}$$

Be careful to distinguish between uses of the product rule and the power rule. When using the product rule, different terms with the same bases are raised to exponents. In this case, you add the exponents. When using the power rule, a term in exponential notation is raised to a power. In this case, you multiply the exponents.

Equation:

$$\begin{array}{rclclcrclcl} & & \text{Product Rule} & & & & & & \text{Power Rule} & & \\ 5^3\cdot 5^4 & = & 5^{3+4} & = & 5^7 & \text{but} & (5^3)^4 & = & 5^{3\cdot 4} & = & 5^{12}\\ x^5\cdot x^2 & = & x^{5+2} & = & x^7 & \text{but} & (x^5)^2 & = & x^{5\cdot 2} & = & x^{10}\\ (3a)^7\cdot(3a)^{10} & = & (3a)^{7+10} & = & (3a)^{17} & \text{but} & ((3a)^7)^{10} & = & (3a)^{7\cdot 10} & = & (3a)^{70}\end{array}$$

Note:

The Power Rule of Exponents

For any real number a and positive integers m and n,the power rule of exponents states that

Equation:

$$(a^m)^n = a^{m\cdot n}$$

Example:

Exercise:

Problem:

Using the Power Rule

Write each of the following products with a single base. Do not simplify further.

a. $\left(x^2\right)^7$

b. $\left((2t)^5\right)^3$

c. $\left((-3)^5\right)^{11}$

Solution:

Use the power rule to simplify each expression.

a. $\left(x^2\right)^7 = x^{2\cdot 7} = x^{14}$
b. $\left((2t)^5\right)^3 = (2t)^{5\cdot 3} = (2t)^{15}$
c. $\left((-3)^5\right)^{11} = (-3)^{5\cdot 11} = (-3)^{55}$

Note:
Exercise:

Problem: Write each of the following products with a single base. Do not simplify further.

a. $\left((3y)^8\right)^3$
b. $\left(t^5\right)^7$
c. $\left((-g)^4\right)^4$

Solution:

a. $(3y)^{24}$
b. t^{35}
c. $(-g)^{16}$

Using the Zero Exponent Rule of Exponents

Return to the quotient rule. We made the condition that $m > n$ so that the difference $m - n$ would never be zero or negative. What would happen if $m = n$?In this case, we would use the *zero exponent rule of exponents* to simplify the expression to 1. To see how this is done, let us begin with an example.
Equation:

$$\frac{t^8}{t^8} = \frac{\cancel{t^8}}{\cancel{t^8}} = 1$$

If we were to simplify the original expression using the quotient rule, we would have
Equation:

$$\frac{t^8}{t^8} = t^{8-8} = t^0$$

If we equate the two answers, the result is $t^0 = 1$. This is true for any nonzero real number, or any variable representing a real number.

Equation:

$$a^0 = 1$$

The sole exception is the expression 0^0. This appears later in more advanced courses, but for now, we will consider the value to be undefined.

Note:
The Zero Exponent Rule of Exponents
For any nonzero real number a, the zero exponent rule of exponents states that
Equation:

$$a^0 = 1$$

Example:
Exercise:

Problem:
Using the Zero Exponent Rule

Simplify each expression using the zero exponent rule of exponents.

a. $\frac{c^3}{c^3}$
b. $\frac{-3x^5}{x^5}$
c. $\frac{\left(j^2k\right)^4}{\left(j^2k\right)\cdot\left(j^2k\right)^3}$
d. $\frac{5\left(rs^2\right)^2}{\left(rs^2\right)^2}$

Solution:

Use the zero exponent and other rules to simplify each expression.

a.
$$\begin{aligned}\frac{c^3}{c^3} &= c^{3-3} \\ &= c^0 \\ &= 1\end{aligned}$$
b.
$$\begin{aligned}\frac{-3x^5}{x^5} &= -3\cdot\frac{x^5}{x^5} \\ &= -3\cdot x^{5-5} \\ &= -3\cdot x^0 \\ &= -3\cdot 1 \\ &= -3\end{aligned}$$

c.

$$\begin{aligned} \frac{(j^2k)^4}{(j^2k)\cdot(j^2k)^3} &= \frac{(j^2k)^4}{(j^2k)^{1+3}} && \text{Use the product rule in the denominator.} \\ &= \frac{(j^2k)^4}{(j^2k)^4} && \text{Simplify.} \\ &= (j^2k)^{4-4} && \text{Use the quotient rule.} \\ &= (j^2k)^0 && \text{Simplify.} \\ &= 1 \end{aligned}$$

d.

$$\begin{aligned} \frac{5(rs^2)^2}{(rs^2)^2} &= 5(rs^2)^{2-2} && \text{Use the quotient rule.} \\ &= 5(rs^2)^0 && \text{Simplify.} \\ &= 5\cdot 1 && \text{Use the zero exponent rule.} \\ &= 5 && \text{Simplify.} \end{aligned}$$

Note:
Exercise:

Problem: Simplify each expression using the zero exponent rule of exponents.

a. $\frac{t^7}{t^7}$
b. $\frac{(de^2)^{11}}{2(de^2)^{11}}$
c. $\frac{w^4\cdot w^2}{w^6}$
d. $\frac{t^3\cdot t^4}{t^2\cdot t^5}$

Solution:

a. 1
b. $\frac{1}{2}$
c. 1
d. 1

Using the Negative Rule of Exponents

Another useful result occurs if we relax the condition that $m > n$ in the quotient rule even further. For example, can we simplify $\frac{h^3}{h^5}$? When $m < n$—that is, where the difference $m - n$ is negative—we can use the *negative rule of exponents* to simplify the expression to its reciprocal.

Divide one exponential expression by another with a larger exponent. Use our example, $\frac{h^3}{h^5}$.
Equation:

$$\begin{aligned} \frac{h^3}{h^5} &= \frac{h \cdot h \cdot h}{h \cdot h \cdot h \cdot h \cdot h} \\ &= \frac{\cancel{h} \cdot \cancel{h} \cdot \cancel{h}}{\cancel{h} \cdot \cancel{h} \cdot \cancel{h} \cdot h \cdot h} \\ &= \frac{1}{h \cdot h} \\ &= \frac{1}{h^2} \end{aligned}$$

If we were to simplify the original expression using the quotient rule, we would have

Equation:

$$\begin{aligned} \frac{h^3}{h^5} &= h^{3-5} \\ &= h^{-2} \end{aligned}$$

Putting the answers together, we have $h^{-2} = \frac{1}{h^2}$. This is true for any nonzero real number, or any variable representing a nonzero real number.

A factor with a negative exponent becomes the same factor with a positive exponent if it is moved across the fraction bar—from numerator to denominator or vice versa.

Equation:

$$a^{-n} = \frac{1}{a^n} \quad \text{and} \quad a^n = \frac{1}{a^{-n}}$$

We have shown that the exponential expression a^n is defined when n is a natural number, 0, or the negative of a natural number. That means that a^n is defined for any integer n. Also, the product and quotient rules and all of the rules we will look at soon hold for any integer n.

Note:

The Negative Rule of Exponents

For any nonzero real number a and natural number n,the negative rule of exponents states that

Equation:

$$a^{-n} = \frac{1}{a^n}$$

Example:

Exercise:

Problem:

Using the Negative Exponent Rule

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

a. $\frac{\theta^3}{\theta^{10}}$

b. $\frac{z^2 \cdot z}{z^4}$

c. $\frac{\left(-5t^3\right)^4}{\left(-5t^3\right)^8}$

Solution:

a. $\frac{\theta^3}{\theta^{10}} = \theta^{3-10} = \theta^{-7} = \frac{1}{\theta^7}$
b. $\frac{z^2 \cdot z}{z^4} = \frac{z^{2+1}}{z^4} = \frac{z^3}{z^4} = z^{3-4} = z^{-1} = \frac{1}{z}$
c. $\frac{\left(-5t^3\right)^4}{\left(-5t^3\right)^8} = \left(-5t^3\right)^{4-8} = \left(-5t^3\right)^{-4} = \frac{1}{\left(-5t^3\right)^4}$

Note:
Exercise:

Problem:

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

a. $\frac{(-3t)^2}{(-3t)^8}$
b. $\frac{f^{47}}{f^{49} \cdot f}$
c. $\frac{2k^4}{5k^7}$

Solution:

a. $\frac{1}{(-3t)^6}$
b. $\frac{1}{f^3}$
c. $\frac{2}{5k^3}$

Example:
Exercise:

Problem:
Using the Product and Quotient Rules

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

a. $b^2 \cdot b^{-8}$
b. $(-x)^5 \cdot (-x)^{-5}$
c. $\frac{-7z}{(-7z)^5}$

Solution:

a. $b^2 \cdot b^{-8} = b^{2-8} = b^{-6} = \frac{1}{b^6}$
b. $(-x)^5 \cdot (-x)^{-5} = (-x)^{5-5} = (-x)^0 = 1$
c. $\frac{-7z}{(-7z)^5} = \frac{(-7z)^1}{(-7z)^5} = (-7z)^{1-5} = (-7z)^{-4} = \frac{1}{(-7z)^4}$

Note:
Exercise:

Problem:

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

a. $t^{-11} \cdot t^{6}$
b. $\frac{25^{12}}{25^{13}}$

Solution:

a. $t^{-5} = \frac{1}{t^5}$
b. $\frac{1}{25}$

Finding the Power of a Product

To simplify the power of a product of two exponential expressions, we can use the *power of a product rule of exponents,* which breaks up the power of a product of factors into the product of the powers of the factors. For instance, consider $(pq)^3$. We begin by using the associative and commutative properties of multiplication to regroup the factors.
Equation:

$$\begin{aligned}(pq)^3 &= \overbrace{(pq)\cdot(pq)\cdot(pq)}^{\text{3 factors}} \\ &= p\cdot q\cdot p\cdot q\cdot p\cdot q \\ &= \overbrace{p\cdot p\cdot p}^{\text{3 factors}}\cdot\overbrace{q\cdot q\cdot q}^{\text{3 factors}} \\ &= p^3\cdot q^3\end{aligned}$$

In other words, $(pq)^3 = p^3 \cdot q^3$.

Note:
The Power of a Product Rule of Exponents
For any real numbers a and b and any integer n,the power of a product rule of exponents states that
Equation:

$$(ab)^n = a^n b^n$$

Example:
Exercise:

Problem:
Using the Power of a Product Rule

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

a. $\left(ab^2\right)^3$
b. $(2t)^{15}$
c. $\left(-2w^3\right)^3$
d. $\frac{1}{(-7z)^4}$
e. $\left(e^{-2}f^2\right)^7$

Solution:

Use the product and quotient rules and the new definitions to simplify each expression.

a. $\left(ab^2\right)^3 = (a)^3 \cdot \left(b^2\right)^3 = a^{1\cdot 3} \cdot b^{2\cdot 3} = a^3b^6$
b. $(2t)^{15} = (2)^{15} \cdot (t)^{15} = 2^{15}t^{15} = 32,768t^{15}$
c. $\left(-2w^3\right)^3 = (-2)^3 \cdot \left(w^3\right)^3 = -8 \cdot w^{3\cdot 3} = -8w^9$
d. $\frac{1}{(-7z)^4} = \frac{1}{(-7)^4 \cdot (z)^4} = \frac{1}{2{,}401z^4}$
e. $\left(e^{-2}f^2\right)^7 = \left(e^{-2}\right)^7 \cdot \left(f^2\right)^7 = e^{-2\cdot 7} \cdot f^{2\cdot 7} = e^{-14}f^{14} = \frac{f^{14}}{e^{14}}$

Note:
Exercise:

Problem:

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

a. $\left(g^2h^3\right)^5$
b. $(5t)^3$
c. $\left(-3y^5\right)^3$
d. $\frac{1}{\left(a^6b^7\right)^3}$
e. $\left(r^3s^{-2}\right)^4$

Solution:

a. $g^{10}h^{15}$
b. $125t^3$
c. $-27y^{15}$
d. $\frac{1}{a^{18}b^{21}}$
e. $\frac{r^{12}}{s^8}$

Finding the Power of a Quotient

To simplify the power of a quotient of two expressions, we can use the *power of a quotient rule*, which states that the power of a quotient of factors is the quotient of the powers of the factors. For example, let's look at the following example.

Equation:

$$\left(e^{-2}f^{2}\right)^{7}=\frac{f^{14}}{e^{14}}$$

Let's rewrite the original problem differently and look at the result.

Equation:

$$\begin{aligned}\left(e^{-2}f^{2}\right)^{7} &= \left(\frac{f^{2}}{e^{2}}\right)^{7}\\ &= \frac{f^{14}}{e^{14}}\end{aligned}$$

It appears from the last two steps that we can use the power of a product rule as a power of a quotient rule.

Equation:

$$\begin{aligned}\left(e^{-2}f^{2}\right)^{7} &= \left(\frac{f^{2}}{e^{2}}\right)^{7}\\ &= \frac{\left(f^{2}\right)^{7}}{\left(e^{2}\right)^{7}}\\ &= \frac{f^{2\cdot 7}}{e^{2\cdot 7}}\\ &= \frac{f^{14}}{e^{14}}\end{aligned}$$

Note:

The Power of a Quotient Rule of Exponents

For any real numbers a and b and any integer n,the power of a quotient rule of exponents states that

Equation:

$$\left(\frac{a}{b}\right)^{n}=\frac{a^{n}}{b^{n}}$$

Example:

Exercise:

Problem:

Using the Power of a Quotient Rule

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

a. $\left(\frac{4}{z^{11}}\right)^{3}$

b. $\left(\frac{p}{q^{3}}\right)^{6}$

c. $\left(\frac{-1}{t^{2}}\right)^{27}$

d. $\left(j^{3}k^{-2}\right)^{4}$

e. $\left(m^{-2}n^{-2}\right)^{3}$

Solution:

a. $\left(\frac{4}{z^{11}}\right)^{3}=\frac{(4)^{3}}{\left(z^{11}\right)^{3}}=\frac{64}{z^{11\cdot 3}}=\frac{64}{z^{33}}$

b. $\left(\frac{p}{q^{3}}\right)^{6}=\frac{(p)^{6}}{\left(q^{3}\right)^{6}}=\frac{p^{1\cdot 6}}{q^{3\cdot 6}}=\frac{p^{6}}{q^{18}}$

c. $\left(\frac{-1}{t^{2}}\right)^{27}=\frac{(-1)^{27}}{\left(t^{2}\right)^{27}}=\frac{-1}{t^{2\cdot 27}}=\frac{-1}{t^{54}}=-\frac{1}{t^{54}}$

d. $\left(j^{3}k^{-2}\right)^{4}=\left(\frac{j^{3}}{k^{2}}\right)^{4}=\frac{\left(j^{3}\right)^{4}}{\left(k^{2}\right)^{4}}=\frac{j^{3\cdot 4}}{k^{2\cdot 4}}=\frac{j^{12}}{k^{8}}$

e. $\left(m^{-2}n^{-2}\right)^{3}=\left(\frac{1}{m^{2}n^{2}}\right)^{3}=\frac{(1)^{3}}{\left(m^{2}n^{2}\right)^{3}}=\frac{1}{\left(m^{2}\right)^{3}\left(n^{2}\right)^{3}}=\frac{1}{m^{2\cdot 3}\cdot n^{2\cdot 3}}=\frac{1}{m^{6}n^{6}}$

Note:
Exercise:

Problem:

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

a. $\left(\frac{b^{5}}{c}\right)^{3}$

b. $\left(\frac{5}{u^{8}}\right)^{4}$

c. $\left(\frac{-1}{w^{3}}\right)^{35}$

d. $\left(p^{-4}q^{3}\right)^{8}$

e. $\left(c^{-5}d^{-3}\right)^{4}$

Solution:

a. $\frac{b^{15}}{c^{3}}$

b. $\frac{625}{u^{32}}$

c. $\frac{-1}{w^{105}}$

d. $\frac{q^{24}}{p^{32}}$

e. $\frac{1}{c^{20}d^{12}}$

Simplifying Exponential Expressions

Recall that to simplify an expression means to rewrite it by combing terms or exponents; in other words, to write the expression more simply with fewer terms. The rules for exponents may be combined to simplify expressions.

Example:
Exercise:

Problem:
Simplifying Exponential Expressions

Simplify each expression and write the answer with positive exponents only.

a. $(6m^2n^{-1})^3$
b. $17^5 \cdot 17^{-4} \cdot 17^{-3}$
c. $\left(\frac{u^{-1}v}{v^{-1}}\right)^2$
d. $(-2a^3b^{-1})(5a^{-2}b^2)$
e. $\left(x^2\sqrt{2}\right)^4\left(x^2\sqrt{2}\right)^{-4}$
f. $\frac{(3w^2)^5}{(6w^{-2})^2}$

Solution:

a.

$$\begin{aligned} (6m^2n^{-1})^3 &= (6)^3(m^2)^3(n^{-1})^3 && \text{The power of a product rule} \\ &= 6^3m^{2\cdot3}n^{-1\cdot3} && \text{The power rule} \\ &= 216m^6n^{-3} && \text{Simplify.} \\ &= \frac{216m^6}{n^3} && \text{The negative exponent rule} \end{aligned}$$

b.

$$\begin{aligned} 17^5 \cdot 17^{-4} \cdot 17^{-3} &= 17^{5-4-3} && \text{The product rule} \\ &= 17^{-2} && \text{Simplify.} \\ &= \frac{1}{17^2} \text{ or } \frac{1}{289} && \text{The negative exponent rule} \end{aligned}$$

c.

$$\begin{aligned} \left(\frac{u^{-1}v}{v^{-1}}\right)^2 &= \frac{(u^{-1}v)^2}{(v^{-1})^2} && \text{The power of a quotient rule} \\ &= \frac{u^{-2}v^2}{v^{-2}} && \text{The power of a product rule} \\ &= u^{-2}v^{2-(-2)} && \text{The quotient rule} \\ &= u^{-2}v^4 && \text{Simplify.} \\ &= \frac{v^4}{u^2} && \text{The negative exponent rule} \end{aligned}$$

d.

$$\begin{aligned} (-2a^3b^{-1})(5a^{-2}b^2) &= -2 \cdot 5 \cdot a^3 \cdot a^{-2} \cdot b^{-1} \cdot b^2 && \text{Commutative and associative laws of mu} \\ &= -10 \cdot a^{3-2} \cdot b^{-1+2} && \text{The product rule} \\ &= -10ab && \text{Simplify.} \end{aligned}$$

e.

$$\begin{aligned} \left(x^2\sqrt{2}\right)^4\left(x^2\sqrt{2}\right)^{-4} &= \left(x^2\sqrt{2}\right)^{4-4} && \text{The product rule} \\ &= \left(x^2\sqrt{2}\right)^0 && \text{Simplify.} \\ &= 1 && \text{The zero exponent rule} \end{aligned}$$

f.

$$\begin{aligned} \frac{(3w^2)^5}{(6w^{-2})^2} &= \frac{(3)^5 \cdot (w^2)^5}{(6)^2 \cdot (w^{-2})^2} && \text{The power of a product rule} \\ &= \frac{3^5 w^{2 \cdot 5}}{6^2 w^{-2 \cdot 2}} && \text{The power rule} \\ &= \frac{243 w^{10}}{36 w^{-4}} && \text{Simplify.} \\ &= \frac{27 w^{10-(-4)}}{4} && \text{The quotient rule and reduce fraction} \\ &= \frac{27 w^{14}}{4} && \text{Simplify.} \end{aligned}$$

Note:
Exercise:

Problem: Simplify each expression and write the answer with positive exponents only.

a. $\left(2uv^{-2}\right)^{-3}$
b. $x^8 \cdot x^{-12} \cdot x$
c. $\left(\frac{e^2 f^{-3}}{f^{-1}}\right)^2$
d. $\left(9r^{-5}s^3\right)\left(3r^6 s^{-4}\right)$
e. $\left(\frac{4}{9}tw^{-2}\right)^{-3}\left(\frac{4}{9}tw^{-2}\right)^3$
f. $\frac{(2h^2k)^4}{(7h^{-1}k^2)^2}$

Solution:

a. $\frac{v^6}{8u^3}$
b. $\frac{1}{x^3}$
c. $\frac{e^4}{f^4}$
d. $\frac{27r}{s}$
e. 1
f. $\frac{16h^{10}}{49}$

Using Scientific Notation

Recall at the beginning of the section that we found the number 1.3×10^{13} when describing bits of information in digital images. Other extreme numbers include the width of a human hair, which is about 0.00005 m, and the radius of an electron, which is about 0.00000000000047 m. How can we effectively work read, compare, and calculate with numbers such as these?

A shorthand method of writing very small and very large numbers is called **scientific notation**, in which we express numbers in terms of exponents of 10. To write a number in scientific notation, move the decimal point to the right of the first digit in the number. Write the digits as a decimal number between 1 and 10. Count the number of places *n* that you moved the decimal point. Multiply the decimal number by 10 raised to a power of *n*. If you moved the decimal left as in a very large number, n is positive. If you moved the decimal right as in a small large number, n is negative.

For example, consider the number 2,780,418. Move the decimal left until it is to the right of the first nonzero digit, which is 2.

$$2{,}780{,}418 \longrightarrow 2.\overbrace{780418}^{\text{6 places left}}$$

We obtain 2.780418 by moving the decimal point 6 places to the left. Therefore, the exponent of 10 is 6, and it is positive because we moved the decimal point to the left. This is what we should expect for a large number.

Equation:

$$2.780418 \times 10^{6}$$

Working with small numbers is similar. Take, for example, the radius of an electron, 0.00000000000047 m. Perform the same series of steps as above, except move the decimal point to the right.

$$0.00000000000047 \longrightarrow 0\overbrace{0000000000004}^{\text{13 places right}}.7$$

Be careful not to include the leading 0 in your count. We move the decimal point 13 places to the right, so the exponent of 10 is 13. The exponent is negative because we moved the decimal point to the right. This is what we should expect for a small number.

Equation:

$$4.7 \times 10^{-13}$$

Note:
Scientific Notation
A number is written in **scientific notation** if it is written in the form $a \times 10^{n}$, where $1 \leq |a| < 10$ and n is an integer.

Example:
Exercise:

Problem:
Converting Standard Notation to Scientific Notation

Write each number in scientific notation.

a. Distance to Andromeda Galaxy from Earth: 24,000,000,000,000,000,000,000 m
b. Diameter of Andromeda Galaxy: 1,300,000,000,000,000,000,000 m
c. Number of stars in Andromeda Galaxy: 1,000,000,000,000
d. Diameter of electron: 0.00000000000094 m
e. Probability of being struck by lightning in any single year: 0.00000143

Solution:

a.
$24{,}000{,}000{,}000{,}000{,}000{,}000{,}000 \text{ m}$
$24{,}000{,}000{,}000{,}000{,}000{,}000{,}000 \text{ m}$
$\leftarrow 22 \text{ places}$
$2.4 \times 10^{22} \text{ m}$
b.
$1{,}300{,}000{,}000{,}000{,}000{,}000{,}000 \text{ m}$
$1{,}300{,}000{,}000{,}000{,}000{,}000{,}000 \text{ m}$
$\leftarrow 21 \text{ places}$
$1.3 \times 10^{21} \text{ m}$
c.
$1{,}000{,}000{,}000{,}000$
$1{,}000{,}000{,}000{,}000$
$\leftarrow 12 \text{ places}$
1×10^{12}
d.
$0.00000000000094 \text{ m}$
$0.00000000000094 \text{ m}$
$\rightarrow 13 \text{ places}$
$9.4 \times 10^{-13} \text{ m}$
e.
0.00000143
0.00000143
$\rightarrow 6 \text{ places}$
1.43×10^{-6}

Analysis

Observe that, if the given number is greater than 1, as in examples a–c, the exponent of 10 is positive; and if the number is less than 1, as in examples d–e, the exponent is negative.

Note:
Exercise:

Problem: Write each number in scientific notation.

a. U.S. national debt per taxpayer (April 2014): $152,000
b. World population (April 2014): 7,158,000,000
c. World gross national income (April 2014): $85,500,000,000,000
d. Time for light to travel 1 m: 0.00000000334 s
e. Probability of winning lottery (match 6 of 49 possible numbers): 0.0000000715

Solution:

a. $\$1.52 \times 10^{5}$
b. 7.158×10^{9}
c. $\$8.55 \times 10^{13}$
d. 3.34×10^{-9}

e. 7.15×10^{-8}

Converting from Scientific to Standard Notation

To convert a number in **scientific notation** to standard notation, simply reverse the process. Move the decimal n places to the right if n is positive or n places to the left if n is negative and add zeros as needed. Remember, if n is positive, the value of the number is greater than 1, and if n is negative, the value of the number is less than one.

Example:
Exercise:

Problem:
Converting Scientific Notation to Standard Notation

Convert each number in scientific notation to standard notation.

a. 3.547×10^{14}
b. -2×10^{6}
c. 7.91×10^{-7}
d. -8.05×10^{-12}

Solution:

a.
$$3.547 \times 10^{14}$$
$$3.54700000000000$$
$$\rightarrow 14 \text{ places}$$
$$354{,}700{,}000{,}000{,}000$$

b.
$$-2 \times 10^{6}$$
$$-2.000000$$
$$\rightarrow 6 \text{ places}$$
$$-2{,}000{,}000$$

c.
$$7.91 \times 10^{-7}$$
$$0000007.91$$
$$\rightarrow 7 \text{ places}$$
$$0.000000791$$

d.
$$-8.05 \times 10^{-12}$$
$$-000000000008.05$$
$$\rightarrow 12 \text{ places}$$
$$-0.0000000000805$$

Note:

Exercise:

Problem: Convert each number in scientific notation to standard notation.

a. 7.03×10^5
b. -8.16×10^{11}
c. -3.9×10^{-13}
d. 8×10^{-6}

Solution:

a. $703,000$
b. $-816,000,000,000$
c. $-0.000\,000\,000\,000\,39$
d. 0.000008

Using Scientific Notation in Applications

Scientific notation, used with the rules of exponents, makes calculating with large or small numbers much easier than doing so using standard notation. For example, suppose we are asked to calculate the number of atoms in 1 L of water. Each water molecule contains 3 atoms (2 hydrogen and 1 oxygen). The average drop of water contains around 1.32×10^{21} molecules of water and 1 L of water holds about 1.22×10^4 average drops. Therefore, there are approximately $3 \cdot (1.32 \times 10^{21}) \cdot (1.22 \times 10^4) \approx 4.83 \times 10^{25}$ atoms in 1 L of water. We simply multiply the decimal terms and add the exponents. Imagine having to perform the calculation without using scientific notation!

When performing calculations with scientific notation, be sure to write the answer in proper scientific notation. For example, consider the product $(7 \times 10^4) \cdot (5 \times 10^6) = 35 \times 10^{10}$. The answer is not in proper scientific notation because 35 is greater than 10. Consider 35 as 3.5×10. That adds a ten to the exponent of the answer.
Equation:

$$(35) \times 10^{10} = (3.5 \times 10) \times 10^{10} = 3.5 \times (10 \times 10^{10}) = 3.5 \times 10^{11}$$

Example:
Exercise:

Problem:
Using Scientific Notation

Perform the operations and write the answer in scientific notation.

a. $(8.14 \times 10^{-7})(6.5 \times 10^{10})$
b. $(4 \times 10^5) \div (-1.52 \times 10^9)$
c. $(2.7 \times 10^5)(6.04 \times 10^{13})$
d. $(1.2 \times 10^8) \div (9.6 \times 10^5)$
e. $(3.33 \times 10^4)(-1.05 \times 10^7)(5.62 \times 10^5)$

Solution:

a.

$$\begin{aligned}\left(8.14 \times 10^{-7}\right)\left(6.5 \times 10^{10}\right) &= (8.14 \times 6.5)\left(10^{-7} \times 10^{10}\right) && \text{Commutative and associativ properties of multiplication} \\ &= (52.91)\left(10^{3}\right) && \text{Product rule of exponents} \\ &= 5.291 \times 10^{4} && \text{Scientific notation}\end{aligned}$$

b.

$$\begin{aligned}\left(4 \times 10^{5}\right) \div \left(-1.52 \times 10^{9}\right) &= \left(\tfrac{4}{-1.52}\right)\left(\tfrac{10^{5}}{10^{9}}\right) && \text{Commutative and associativ properties of multiplication} \\ &\approx (-2.63)\left(10^{-4}\right) && \text{Quotient rule of exponents} \\ &= -2.63 \times 10^{-4} && \text{Scientific notation}\end{aligned}$$

c.

$$\begin{aligned}\left(2.7 \times 10^{5}\right)\left(6.04 \times 10^{13}\right) &= (2.7 \times 6.04)\left(10^{5} \times 10^{13}\right) && \text{Commutative and associativ properties of multiplication} \\ &= (16.308)\left(10^{18}\right) && \text{Product rule of exponents} \\ &= 1.6308 \times 10^{19} && \text{Scientific notation}\end{aligned}$$

d.

$$\begin{aligned}\left(1.2 \times 10^{8}\right) \div \left(9.6 \times 10^{5}\right) &= \left(\tfrac{1.2}{9.6}\right)\left(\tfrac{10^{8}}{10^{5}}\right) && \text{Commutative and associative properties of multiplication} \\ &= (0.125)\left(10^{3}\right) && \text{Quotient rule of exponents} \\ &= 1.25 \times 10^{2} && \text{Scientific notation}\end{aligned}$$

e.

$$\begin{aligned}\left(3.33 \times 10^{4}\right)\left(-1.05 \times 10^{7}\right)\left(5.62 \times 10^{5}\right) &= [3.33 \times (-1.05) \times 5.62]\left(10^{4} \times 10^{7} \times 10^{5}\right) \\ &\approx (-19.65)\left(10^{16}\right) \\ &= -1.965 \times 10^{17}\end{aligned}$$

Note:
Exercise:

Problem: Perform the operations and write the answer in scientific notation.

a. $\left(-7.5 \times 10^{8}\right)\left(1.13 \times 10^{-2}\right)$
b. $\left(1.24 \times 10^{11}\right) \div \left(1.55 \times 10^{18}\right)$
c. $\left(3.72 \times 10^{9}\right)\left(8 \times 10^{3}\right)$
d. $\left(9.933 \times 10^{23}\right) \div \left(-2.31 \times 10^{17}\right)$
e. $\left(-6.04 \times 10^{9}\right)\left(7.3 \times 10^{2}\right)\left(-2.81 \times 10^{2}\right)$

Solution:

a. -8.475×10^{6}
b. 8×10^{-8}
c. 2.976×10^{13}
d. -4.3×10^{6}
e. $\approx 1.24 \times 10^{15}$

Example:
Exercise:

Problem:
Applying Scientific Notation to Solve Problems

In April 2014, the population of the United States was about 308,000,000 people. The national debt was about $17,547,000,000,000. Write each number in scientific notation, rounding figures to two decimal places, and find the amount of the debt per U.S. citizen. Write the answer in both scientific and standard notations.

Solution:

The population was $308{,}000{,}000 = 3.08 \times 10^8$.

The national debt was $\$17{,}547{,}000{,}000{,}000 \approx \$1.75 \times 10^{13}$.

To find the amount of debt per citizen, divide the national debt by the number of citizens.
Equation:

$$\begin{aligned}\left(1.75 \times 10^{13}\right) \div \left(3.08 \times 10^{8}\right) &= \left(\frac{1.75}{3.08}\right) \cdot \left(\frac{10^{13}}{10^{8}}\right) \\ &\approx 0.57 \times 10^5 \\ &= 5.7 \times 10^4\end{aligned}$$

The debt per citizen at the time was about $\$5.7 \times 10^4$,or $57,000.

Note:
Exercise:

Problem:

An average human body contains around 30,000,000,000,000 red blood cells. Each cell measures approximately 0.000008 m long. Write each number in scientific notation and find the total length if the cells were laid end-to-end. Write the answer in both scientific and standard notations.

Solution:

Number of cells: 3×10^{13};length of a cell: 8×10^{-6} m; total length: 2.4×10^8 m or $240{,}000{,}000$ m.

Note:
Access these online resources for additional instruction and practice with exponents and scientific notation.

- Exponential Notation
- Properties of Exponents
- Zero Exponent
- Simplify Exponent Expressions
- Quotient Rule for Exponents
- Scientific Notation
- Converting to Decimal Notation

Key Equations

Rules of Exponents For nonzero real numbers a and b and integers m and n	
Product rule	$a^m \cdot a^n = a^{m+n}$
Quotient rule	$\frac{a^m}{a^n} = a^{m-n}$
Power rule	$(a^m)^n = a^{m \cdot n}$
Zero exponent rule	$a^0 = 1$
Negative rule	$a^{-n} = \frac{1}{a^n}$
Power of a product rule	$(a \cdot b)^n = a^n \cdot b^n$
Power of a quotient rule	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Key Concepts

- Products of exponential expressions with the same base can be simplified by adding exponents. See [link].
- Quotients of exponential expressions with the same base can be simplified by subtracting exponents. See [link].
- Powers of exponential expressions with the same base can be simplified by multiplying exponents. See [link].
- An expression with exponent zero is defined as 1. See [link].
- An expression with a negative exponent is defined as a reciprocal. See [link] and [link].
- The power of a product of factors is the same as the product of the powers of the same factors. See [link].
- The power of a quotient of factors is the same as the quotient of the powers of the same factors. See [link].
- The rules for exponential expressions can be combined to simplify more complicated expressions. See [link].
- Scientific notation uses powers of 10 to simplify very large or very small numbers. See [link] and [link].
- Scientific notation may be used to simplify calculations with very large or very small numbers. See [link] and [link].

Section Exercises

Verbal

Exercise:

Problem: Is 2^3 the same as 3^2? Explain.

Solution:

No, the two expressions are not the same. An exponent tells how many times you multiply the base. So 2^3 is the same as $2 \times 2 \times 2$,which is 8. 3^2 is the same as 3×3,which is 9.

Exercise:

Problem: When can you add two exponents?

Exercise:

Problem: What is the purpose of scientific notation?

Solution:

It is a method of writing very small and very large numbers.

Exercise:

Problem: Explain what a negative exponent does.

Numeric

For the following exercises, simplify the given expression. Write answers with positive exponents.

Exercise:

Problem: 9^2

Solution:

81

Exercise:

Problem: 15^{-2}

Exercise:

Problem: $3^2 \times 3^3$

Solution:

243

Exercise:

Problem: $4^4 \div 4$

Exercise:

Problem: $\left(2^2\right)^{-2}$

Solution:

$\frac{1}{16}$

Exercise:

Problem: $(5-8)^0$

Exercise:

Problem: $11^3 \div 11^4$

Solution:

$\frac{1}{11}$

Exercise:

Problem: $6^5 \times 6^{-7}$

Exercise:

Problem: $\left(8^0\right)^2$

Solution:

1

Exercise:

Problem: $5^{-2} \div 5^2$

For the following exercises, write each expression with a single base. Do not simplify further. Write answers with positive exponents.

Exercise:

Problem: $4^2 \times 4^3 \div 4^{-4}$

Solution:

4^9

Exercise:

Problem: $\frac{6^{12}}{6^9}$

Exercise:

Problem: $\left(12^3 \times 12\right)^{10}$

Solution:

12^{40}

Exercise:

Problem: $10^6 \div \left(10^{10}\right)^{-2}$

Exercise:

Problem: $7^{-6} \times 7^{-3}$

Solution:

$\frac{1}{7^9}$

Exercise:

Problem: $\left(3^3 \div 3^4\right)^5$

For the following exercises, express the decimal in scientific notation.

Exercise:

Problem: 0.0000314

Solution:

3.14×10^{-5}

Exercise:

Problem: 148,000,000

For the following exercises, convert each number in scientific notation to standard notation.

Exercise:

Problem: 1.6×10^{10}

Solution:

16,000,000,000

Exercise:

Problem: 9.8×10^{-9}

Algebraic

For the following exercises, simplify the given expression. Write answers with positive exponents.

Exercise:

Problem: $\frac{a^3a^2}{a}$

Solution:

a^4

Exercise:

Problem: $\frac{mn^2}{m^{-2}}$

Exercise:

Problem: $\left(b^3c^4\right)^2$

Solution:

b^6c^8

Exercise:

Problem: $\left(\frac{x^{-3}}{y^2}\right)^{-5}$

Exercise:

Problem: $ab^2 \div d^{-3}$

Solution:

ab^2d^3

Exercise:

Problem: $\left(w^0x^5\right)^{-1}$

Exercise:

Problem: $\frac{m^4}{n^0}$

Solution:

m^4

Exercise:

Problem: $y^{-4}\left(y^2\right)^2$

Exercise:

Problem: $\frac{p^{-4}q^2}{p^2q^{-3}}$

Solution:

$\frac{q^5}{p^6}$

Exercise:

Problem: $(l \ \times \ w)^2$

Exercise:

Problem: $\left(y^7\right)^3 \div x^{14}$

Solution:

$\frac{y^{21}}{x^{14}}$

Exercise:

Problem: $\left(\frac{a}{2^3}\right)^2$

Exercise:

Problem: $5^2m \div 5^0m$

Solution:

25

Exercise:

Problem: $\frac{\left(16\sqrt{x}\right)^2}{y^{-1}}$

Exercise:

Problem: $\frac{2^3}{(3a)^{-2}}$

Solution:

$72a^2$

Exercise:

Problem: $\left(ma^6\right)^2 \frac{1}{m^3a^2}$

Exercise:

Problem: $\left(b^{-3}c\right)^3$

Solution:

$\frac{c^3}{b^9}$

Exercise:

Problem: $\left(x^2y^{13} \div y^0\right)^2$

Exercise:

Problem: $\left(9z^3\right)^{-2}y$

Solution:

$\frac{y}{81z^6}$

Real-World Applications

Exercise:

Problem:

To reach escape velocity, a rocket must travel at the rate of 2.2×10^6 ft/min. Rewrite the rate in standard notation.

Exercise:

Problem:

A dime is the thinnest coin in U.S. currency. A dime's thickness measures 1.35×10^{-3} m. Rewrite the number in standard notation.

Solution:

0.00135 m

Exercise:

Problem:

The average distance between Earth and the Sun is 92,960,000 mi. Rewrite the distance using scientific notation.

Exercise:

Problem:A terabyte is made of approximately 1,099,500,000,000 bytes. Rewrite in scientific notation.

Solution:

1.0995×10^{12}

Exercise:

Problem:

The Gross Domestic Product (GDP) for the United States in the first quarter of 2014 was $\$1.71496 \times 10^{13}$. Rewrite the GDP in standard notation.

Exercise:

Problem: One picometer is approximately 3.397×10^{-11} in. Rewrite this length using standard notation.

Solution:

0.00000000003397 in.

Exercise:

Problem:

The value of the services sector of the U.S. economy in the first quarter of 2012 was $10,633.6 billion. Rewrite this amount in scientific notation.

Technology

For the following exercises, use a graphing calculator to simplify. Round the answers to the nearest hundredth.

Exercise:

Problem:$\left(\frac{12^3 m^{33}}{4^{-3}}\right)^2$

Solution:

12,230,590,464 m^{66}

Exercise:

Problem: $17^3 \div 15^2 x^3$

Extensions

For the following exercises, simplify the given expression. Write answers with positive exponents.

Exercise:

Problem: $\left(\frac{3^2}{a^3}\right)^{-2}\left(\frac{a^4}{2^2}\right)^2$

Solution:

$\frac{a^{14}}{1296}$

Exercise:

Problem: $\left(6^2 - 24\right)^2 \div \left(\frac{x}{y}\right)^{-5}$

Exercise:

Problem: $\frac{m^2n^3}{a^2c^{-3}} \cdot \frac{a^{-7}n^{-2}}{m^2c^4}$

Solution:

$\frac{n}{a^9c}$

Exercise:

Problem: $\left(\frac{x^6y^3}{x^3y^{-3}} \cdot \frac{y^{-7}}{x^{-3}}\right)^{10}$

Exercise:

Problem: $\left(\frac{\left(ab^2c\right)^{-3}}{b^{-3}}\right)^2$

Solution:

$\frac{1}{a^6b^6c^6}$

Exercise:

Problem:

Avogadro’s constant is used to calculate the number of particles in a mole. A mole is a basic unit in chemistry to measure the amount of a substance. The constant is 6.0221413×10^{23}. Write Avogadro’s constant in standard notation.

Exercise:

Problem:

Planck’s constant is an important unit of measure in quantum physics. It describes the relationship between energy and frequency. The constant is written as $6.62606957 \times 10^{-34}$. Write Planck’s constant in standard notation.

Solution:

0.000000000000000000000000000000000662606957

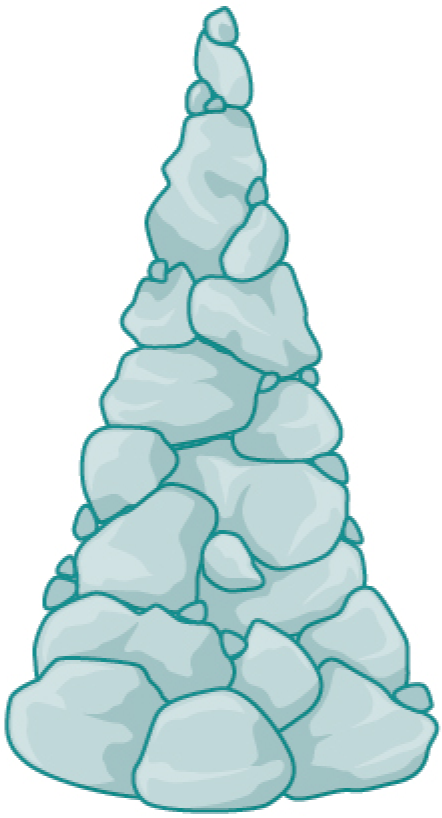
Glossary

scientific notation
: a shorthand notation for writing very large or very small numbers in the form $a \times 10^{n}$ where $1 \leq |a| < 10$ and n is an integer

Inverses and Radical Functions
In this section, you will:

- Find the inverse of a polynomial function.
- Restrict the domain to find the inverse of a polynomial function.

A mound of gravel is in the shape of a cone with the height equal to twice the radius.



The volume is found using a formula from elementary geometry.
Equation:

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi r^2(2r) \\ &= \frac{2}{3}\pi r^3 \end{aligned}$$

We have written the volume V in terms of the radius r. However, in some cases, we may start out with the volume and want to find the radius. For

example: A customer purchases 100 cubic feet of gravel to construct a cone shape mound with a height twice the radius. What are the radius and height of the new cone? To answer this question, we use the formula
Equation:

$$r = \sqrt[3]{\frac{3V}{2\pi}}$$

This function is the inverse of the formula for V in terms of r.

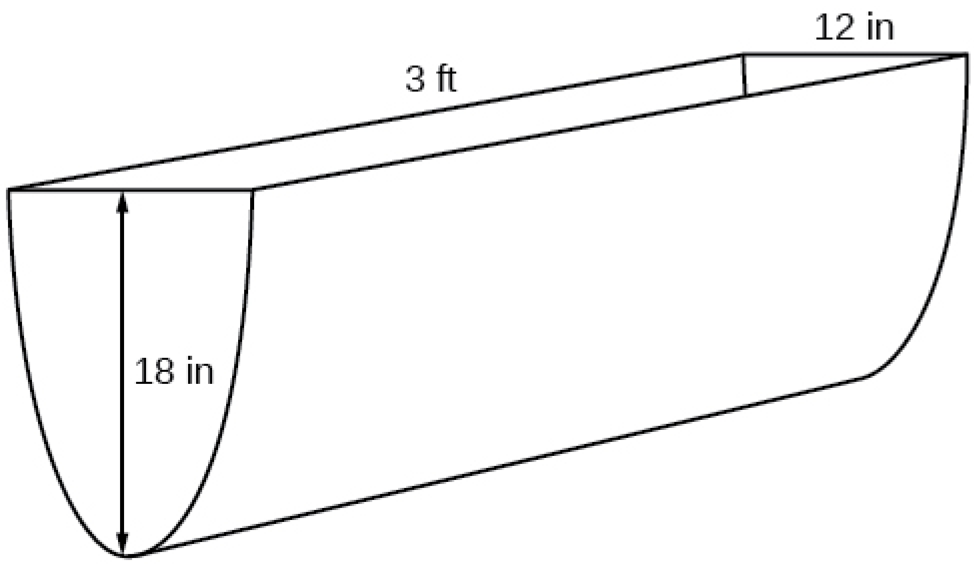
In this section, we will explore the inverses of polynomial and rational functions and in particular the radical functions we encounter in the process.

Finding the Inverse of a Polynomial Function

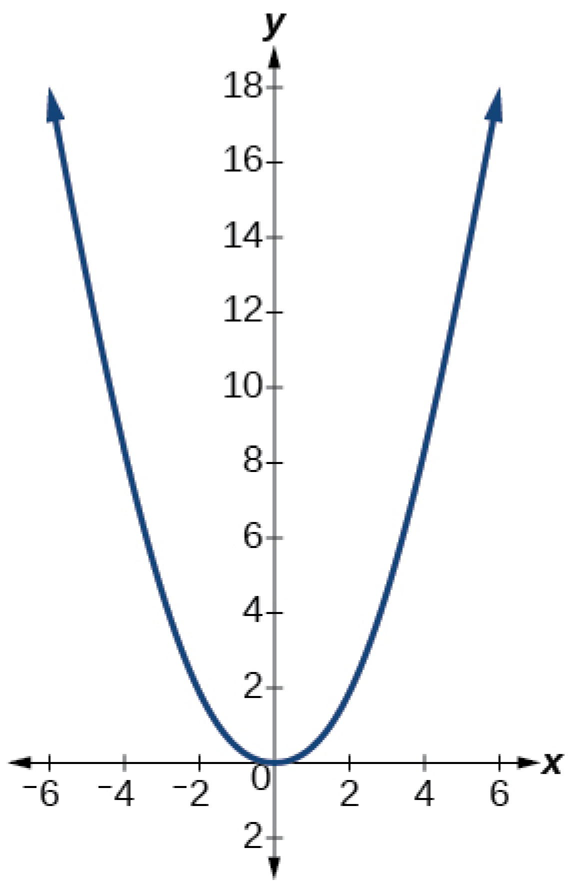
Two functions f and g are inverse functions if for every coordinate pair in $f,\ (a,\ b)$, there exists a corresponding coordinate pair in the inverse function, $g,\ (b,\ a)$. In other words, the coordinate pairs of the inverse functions have the input and output interchanged.

For a function to have an inverse function the function to create a new function that is one-to-one and would have an inverse function.

For example, suppose a water runoff collector is built in the shape of a parabolic trough as shown in [link]. We can use the information in the figure to find the surface area of the water in the trough as a function of the depth of the water.



Because it will be helpful to have an equation for the parabolic cross-sectional shape, we will impose a coordinate system at the cross section, with x measured horizontally and y measured vertically, with the origin at the vertex of the parabola. See [link].



From this we find an equation for the parabolic shape. We placed the origin at the vertex of the parabola, so we know the equation will have form $y(x) = ax^2$. Our equation will need to pass through the point (6, 18), from which we can solve for the stretch factor a.

Equation:

$$\begin{aligned} 18 &= a6^2 \\ a &= \frac{18}{36} \\ &= \frac{1}{2} \end{aligned}$$

Our parabolic cross section has the equation

Equation:

$$y(x) = \frac{1}{2}x^2$$

We are interested in the surface area of the water, so we must determine the width at the top of the water as a function of the water depth. For any depth y the width will be given by $2x$, so we need to solve the equation above for x and find the inverse function. However, notice that the original function is not one-to-one, and indeed, given any output there are two inputs that produce the same output, one positive and one negative.

To find an inverse, we can restrict our original function to a limited domain on which it *is* one-to-one. In this case, it makes sense to restrict ourselves to positive x values. On this domain, we can find an inverse by solving for the input variable:
Equation:

$$\begin{aligned} y &= \tfrac{1}{2}x^2 \\ 2y &= x^2 \\ x &= \pm\sqrt{2y} \end{aligned}$$

This is not a function as written. We are limiting ourselves to positive x values, so we eliminate the negative solution, giving us the inverse function we're looking for.
Equation:

$$y = \frac{x^2}{2},\ x > 0$$

Because x is the distance from the center of the parabola to either side, the entire width of the water at the top will be $2x$. The trough is 3 feet (36 inches) long, so the surface area will then be:
Equation:

$$\begin{aligned} \text{Area} &= l \cdot w \\ &= 36 \cdot 2x \\ &= 72x \\ &= 72\sqrt{2y} \end{aligned}$$

This example illustrates two important points:

1. When finding the inverse of a quadratic, we have to limit ourselves to a domain on which the function is one-to-one.
2. The inverse of a quadratic function is a square root function. Both are toolkit functions and different types of power functions.

Functions involving roots are often called radical functions. While it is not possible to find an inverse of most polynomial functions, some basic polynomials do have inverses. Such functions are called **invertible functions**, and we use the notation $f^{-1}(x)$.

Warning: $f^{-1}(x)$ is not the same as the reciprocal of the function $f(x)$. This use of "–1" is reserved to denote inverse functions. To denote the reciprocal of a function $f(x)$, we would need to write $(f(x))^{-1} = \frac{1}{f(x)}$.

An important relationship between inverse functions is that they "undo" each other. If f^{-1} is the inverse of a function f, then f is the inverse of the function f^{-1}. In other words, whatever the function f does to x, f^{-1} undoes it—and vice-versa. More formally, we write

Equation:

$$f^{-1}(f(x)) = x, \text{ for all } x \text{ in the domain of } f$$

and

Equation:

$$f\left(f^{-1}(x)\right) = x, \text{ for all } x \text{ in the domain of } f^{-1}$$

Note:
Verifying Two Functions Are Inverses of One Another
Two functions, f and g, are inverses of one another if for all x in the domain of f and g.
Equation:

$$g\left(f\left(x\right)\right)=f\left(g\left(x\right)\right)=x$$

Note:
Given a polynomial function, find the inverse of the function by restricting the domain in such a way that the new function is one-to-one.

1. Replace $f\left(x\right)$ with y.
2. Interchange x and y.
3. Solve for y, and rename the function $f^{-1}(x)$.

Example:
Exercise:

Problem:
Verifying Inverse Functions

Show that $f\left(x\right)=\frac{1}{x+1}$ and $f^{-1}\left(x\right)=\frac{1}{x}-1$ are inverses, for $x\neq 0,-1$.

Solution:

We must show that $f^{-1}\left(f\left(x\right)\right)=x$ and $f\left(f^{-1}\left(x\right)\right)=x$.
Equation:

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}\left(\frac{1}{x+1}\right) \\ &= \frac{1}{\frac{1}{x+1}} - 1 \\ &= (x+1) - 1 \\ &= x \\ f(f^{-1}(x)) &= f\left(\frac{1}{x} - 1\right) \\ &= \frac{1}{\left(\frac{1}{x}-1\right)+1} \\ &= \frac{1}{\frac{1}{x}} \\ &= x \end{aligned}$$

Therefore, $f\left(x\right) = \frac{1}{x+1}$ and $f^{-1}\left(x\right) = \frac{1}{x} - 1$ are inverses.

Note:
Exercise:

Problem: Show that $f\left(x\right) = \frac{x+5}{3}$ and $f^{-1}\left(x\right) = 3x - 5$ are inverses.

Solution:

$f^{-1}\left(f\left(x\right)\right) = f^{-1}\left(\frac{x+5}{3}\right) = 3\left(\frac{x+5}{3}\right) - 5 = (x-5) + 5 = x$ and
$f\left(f^{-1}\left(x\right)\right) = f\left(3x-5\right) = \frac{(3x-5)+5}{3} = \frac{3x}{3} = x$

Example:
Exercise:

Problem:
Finding the Inverse of a Cubic Function

Find the inverse of the function $f(x) = 5x^3 + 1$.

Solution:

This is a transformation of the basic cubic toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for x.

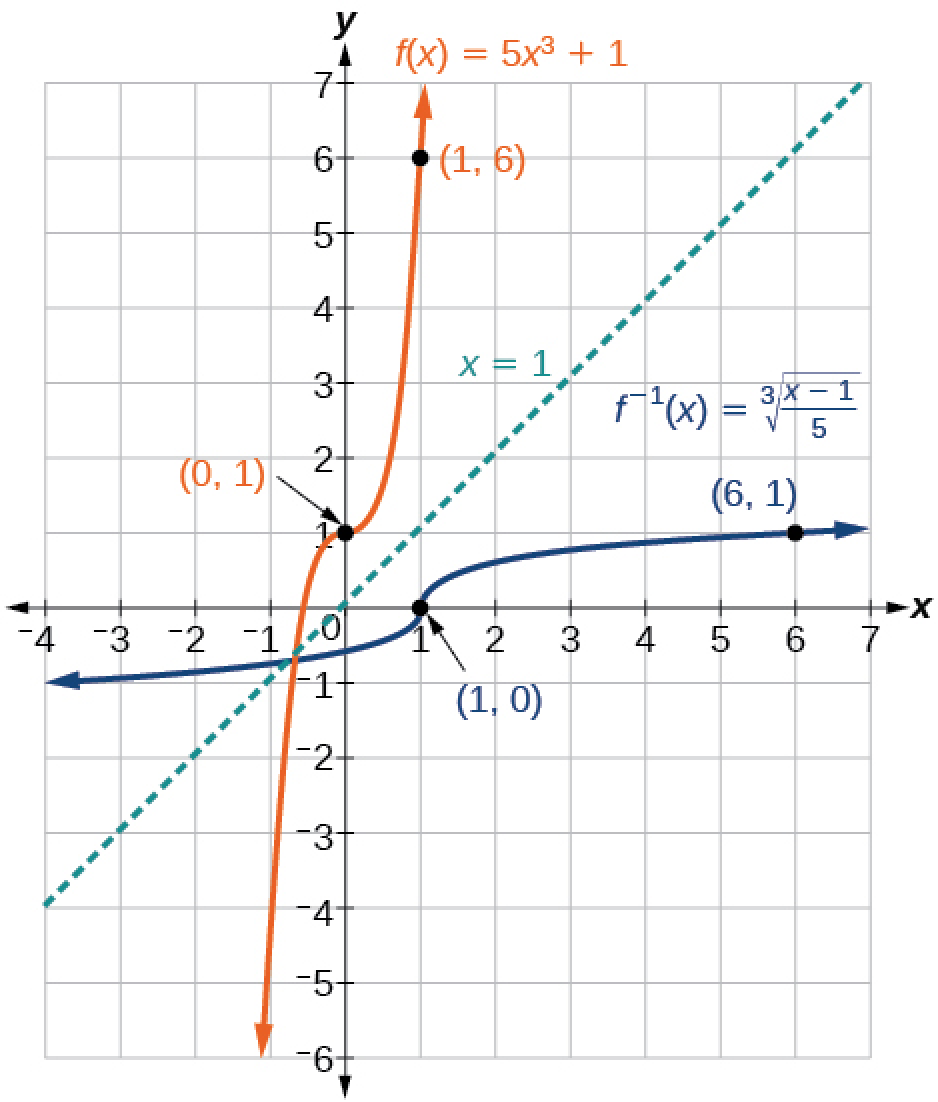
Equation:

$$\begin{aligned} y &= 5x^3 + 1 \\ x &= 5y^3 + 1 \\ x - 1 &= 5y^3 \\ \tfrac{x-1}{5} &= y^3 \\ f^{-1}(x) &= \sqrt[3]{\tfrac{x-1}{5}} \end{aligned}$$

Analysis

Look at the graph of f and f^{-1}. Notice that the two graphs are symmetrical about the line $y = x$. This is always the case when graphing a function and its inverse function.

Also, since the method involved interchanging x and y, notice corresponding points. If (a, b) is on the graph of f,then (b, a) is on the graph of f^{-1}. Since $(0, 1)$ is on the graph of f, then $(1, 0)$ is on the graph of f^{-1}. Similarly, since $(1, 6)$ is on the graph of f,then $(6, 1)$ is on the graph of f^{-1}. See [link].



Note:

Exercise:

Problem: Find the inverse function of $f(x) = \sqrt[3]{x+4}$.

Solution:

$f^{-1}(x) = x^3 - 4$

Restricting the Domain to Find the Inverse of a Polynomial Function

So far, we have been able to find the inverse functions of cubic functions without having to restrict their domains. However, as we know, not all cubic polynomials are one-to-one. Some functions that are not one-to-one may have their domain restricted so that they are one-to-one, but only over that domain. The function over the restricted domain would then have an inverse function. Since quadratic functions are not one-to-one, we must restrict their domain in order to find their inverses.

Note:
Restricting the Domain
If a function is not one-to-one, it cannot have an inverse. If we restrict the domain of the function so that it becomes one-to-one, thus creating a new function, this new function will have an inverse.

Note:
Given a polynomial function, restrict the domain of a function that is not one-to-one and then find the inverse.

1. Restrict the domain by determining a domain on which the original function is one-to-one.
2. Replace $f(x)$ with y.
3. Interchange x and y.
4. Solve for y, and rename the function or pair of function $f^{-1}(x)$.
5. Revise the formula for $f^{-1}(x)$ by ensuring that the outputs of the inverse function correspond to the restricted domain of the original function.

Example:
Exercise:

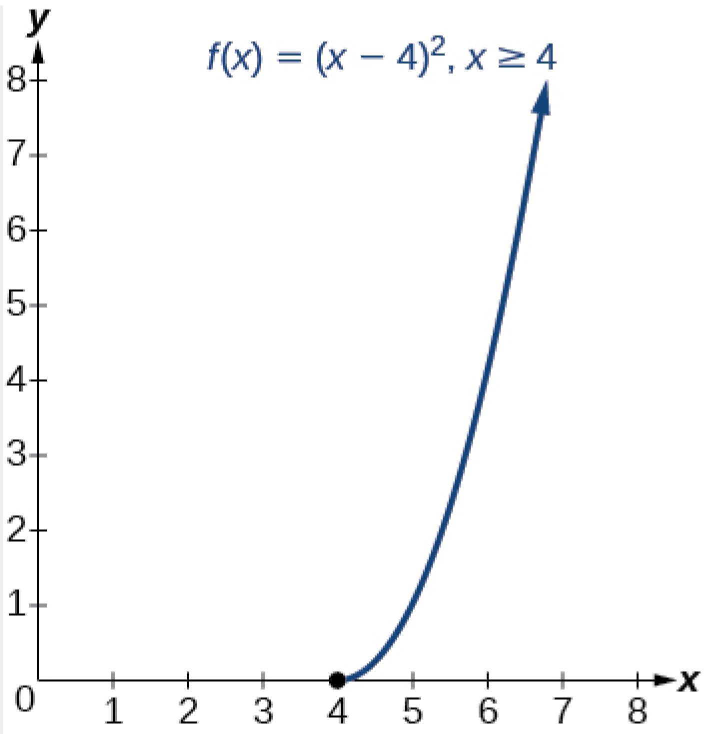
Problem:
Restricting the Domain to Find the Inverse of a Polynomial Function

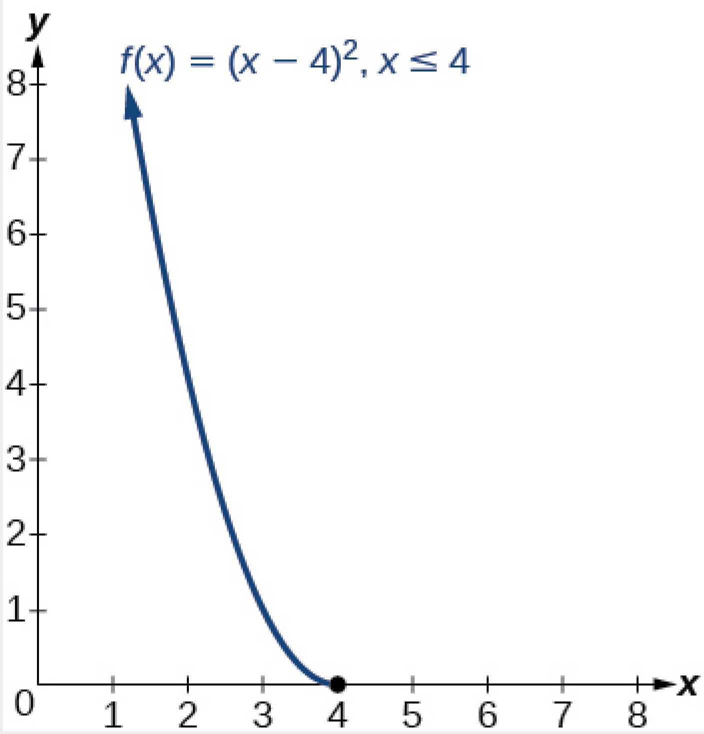
Find the inverse function of f :

a. $f(x) = (x - 4)^2,\ x \geq 4$
b. $f(x) = (x - 4)^2,\ x \leq 4$

Solution:

The original function $f(x) = (x - 4)^2$ is not one-to-one, but the function is restricted to a domain of $x \geq 4$ or $x \leq 4$ on which it is one-to-one. See [link].





To find the inverse, start by replacing $f(x)$ with the simple variable y.
Equation:

$$\begin{aligned} y &= (x-4)^2 && \text{Interchange } x \text{ and } y. \\ x &= (y-4)^2 && \text{Take the square root.} \\ \pm\sqrt{x} &= y-4 && \text{Add 4 to both sides.} \\ 4 \pm \sqrt{x} &= y \end{aligned}$$

This is not a function as written. We need to examine the restrictions on the domain of the original function to determine the inverse. Since we reversed the roles of x and y for the original $f(x)$, we looked at the domain: the values x could assume. When we reversed the roles of x and y, this gave us the values y could assume. For this function, $x \geq 4$, so for the inverse, we should have $y \geq 4$, which is what our inverse function gives.

a. The domain of the original function was restricted to $x \geq 4$, so the outputs of the inverse need to be the same, $f(x) \geq 4$, and we must use the + case:
   Equation:

$$f^{-1}(x) = 4 + \sqrt{x}$$

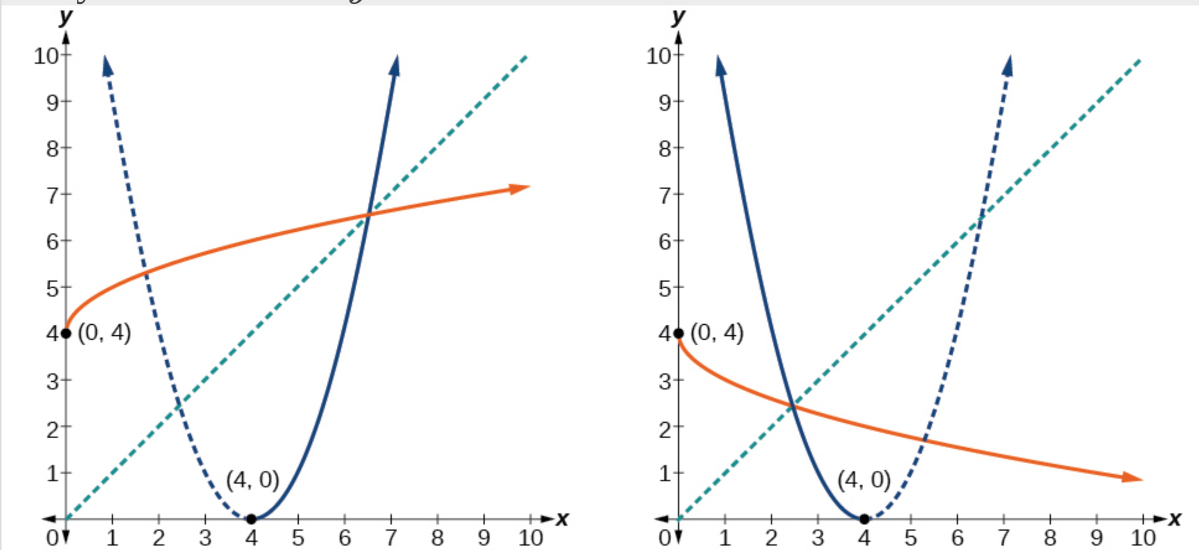
b. The domain of the original function was restricted to $x \leq 4$, so the outputs of the inverse need to be the same, $f(x) \leq 4$, and we must use the – case:
   Equation:

$$f^{-1}(x) = 4 - \sqrt{x}$$

Analysis

On the graphs in [link], we see the original function graphed on the same set of axes as its inverse function. Notice that together the graphs show symmetry about the line $y = x$. The coordinate pair $(4,\ 0)$ is on the graph of f and the coordinate pair $(0,\ 4)$ is on the graph of f^{-1}. For any coordinate pair, if $(a,\ b)$ is on the graph of f, then $(b,\ a)$ is on the graph

of f^{-1}. Finally, observe that the graph of f intersects the graph of f^{-1} on the line $y = x$. Points of intersection for the graphs of f and f^{-1} will always lie on the line $y = x$.



Example:
Exercise:

Problem:
Finding the Inverse of a Quadratic Function When the Restriction Is Not Specified

Restrict the domain and then find the inverse of
Equation:

$$f(x) = (x - 2)^2 - 3.$$

Solution:

We can see this is a parabola with vertex at $(2,\ -3)$ that opens upward. Because the graph will be decreasing on one side of the vertex and increasing on the other side, we can restrict this function to a domain on which it will be one-to-one by limiting the domain to $x \geq 2$.

To find the inverse, we will use the vertex form of the quadratic. We start by replacing $f(x)$ with a simple variable, $y,$ then solve for $x.$

Equation:

$$\begin{array}{rll} y = (x-2)^2 - 3 & & \text{Interchange } x \text{ and } y. \\ x = (y-2)^2 - 3 & & \text{Add 3 to both sides.} \\ x + 3 = (y-2)^2 & & \text{Take the square root.} \\ \pm\sqrt{x+3} = y - 2 & & \text{Add 2 to both sides.} \\ 2 \pm \sqrt{x+3} = y & & \text{Rename the function.} \\ f^{-1}(x) = 2 \pm \sqrt{x+3} & & \end{array}$$

Now we need to determine which case to use. Because we restricted our original function to a domain of $x \ge 2,$ the outputs of the inverse should be the same, telling us to utilize the + case

Equation:

$$f^{-1}(x) = 2 + \sqrt{x+3}$$

If the quadratic had not been given in vertex form, rewriting it into vertex form would be the first step. This way we may easily observe the coordinates of the vertex to help us restrict the domain.

Analysis

Notice that we arbitrarily decided to restrict the domain on $x \ge 2.$ We could just have easily opted to restrict the domain on $x \le 2,$ in which case $f^{-1}(x) = 2 - \sqrt{x+3}.$ Observe the original function graphed on the same set of axes as its inverse function in [link]. Notice that both graphs show symmetry about the line $y = x.$ The coordinate pair $(2,\ -3)$ is on the graph of f and the coordinate pair $(-3,\ 2)$ is on the graph of $f^{-1}.$ Observe from the graph of both functions on the same set of axes that

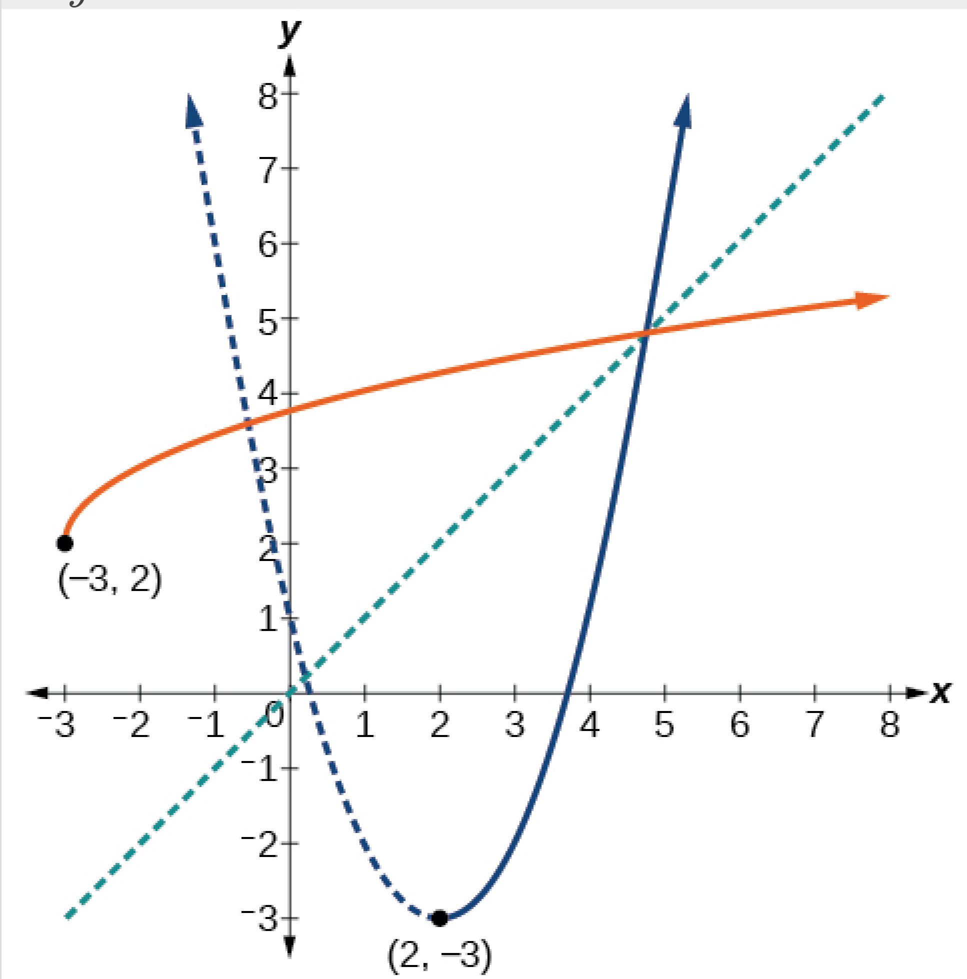
Equation:

$$\text{domain of } f = \text{range of } f^{-1} = [2, \infty)$$

and
Equation:

$$\text{domain of } f^{-1} = \text{range of } f = [-3, \infty)$$

Finally, observe that the graph of f intersects the graph of f^{-1} along the line $y = x$.



Note:
Exercise:

Problem:

Find the inverse of the function $f(x) = x^2 + 1,$ on the domain $x \geq 0$.

Solution:

$f^{-1}(x) = \sqrt{x-1}$

Solving Applications of Radical Functions

Notice that the functions from previous examples were all polynomials, and their inverses were radical functions. If we want to find the inverse of a radical function, we will need to restrict the domain of the answer because the range of the original function is limited.

Note:
Given a radical function, find the inverse.

1. Determine the range of the original function.
2. Replace $f\left(x\right)$ with $y,$ then solve for $x.$
3. If necessary, restrict the domain of the inverse function to the range of the original function.

Example:
Exercise:

Problem:
Finding the Inverse of a Radical Function

Restrict the domain and then find the inverse of the function $f(x) = \sqrt{x-4}$.

Solution:

Note that the original function has range $f(x) \geq 0$. Replace $f(x)$ with y, then solve for x.

Equation:

$$\begin{aligned} y &= \sqrt{x-4} && \text{Replace} f(x) \text{with} y. \\ x &= \sqrt{y-4} && \text{Interchange} x \text{and} y. \\ x &= \sqrt{y-4} && \text{Square each side.} \\ x^2 &= y-4 && \text{Add 4.} \\ x^2+4 &= y && \text{Rename the function} f^{-1}(x). \\ f^{-1}(x) &= x^2+4 \end{aligned}$$

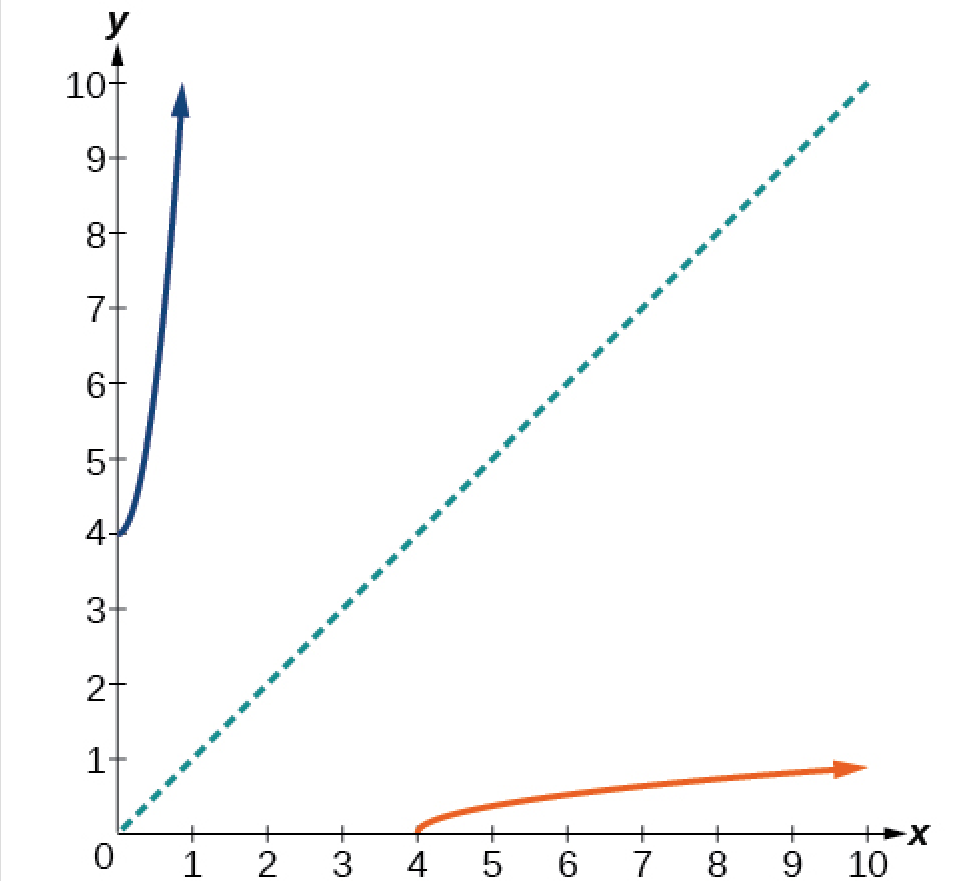
Recall that the domain of this function must be limited to the range of the original function.

Equation:

$$f^{-1}(x) = x^2+4, x \geq 0$$

Analysis

Notice in [link] that the inverse is a reflection of the original function over the line $y = x$. Because the original function has only positive outputs, the inverse function has only positive inputs.



Note:
Exercise:

Problem:

Restrict the domain and then find the inverse of the function $f(x) = \sqrt{2x + 3}$.

Solution:

$f^{-1}(x) = \frac{x^2 - 3}{2}, x \geq 0$

Radical functions are common in physical models, as we saw in the section opener. We now have enough tools to be able to solve the problem posed at the start of the section.

Example:
Exercise:

Problem:
Solving an Application with a Cubic Function

A mound of gravel is in the shape of a cone with the height equal to twice the radius. The volume of the cone in terms of the radius is given by
Equation:

$$V=\frac{2}{3}\pi r^3$$

Find the inverse of the function $V=\frac{2}{3}\pi r^3$ that determines the volume V of a cone and is a function of the radius $r.$ Then use the inverse function to calculate the radius of such a mound of gravel measuring 100 cubic feet. Use $\pi=3.14.$

Solution:

Start with the given function for $V.$ Notice that the meaningful domain for the function is $r\geq 0$ since negative radii would not make sense in this context. Also note the range of the function (hence, the domain of the inverse function) is $V\geq 0.$ Solve for r in terms of $V,$ using the method outlined previously.
Equation:

$$\begin{aligned} V &= \tfrac{2}{3}\pi r^3 & \\ r^3 &= \tfrac{3V}{2\pi} & \text{Solve for } r^3. \\ r &= \sqrt[3]{\tfrac{3V}{2\pi}} & \text{Solve for } r. \end{aligned}$$

This is the result stated in the section opener. Now evaluate this for $V = 100$ and $\pi = 3.14$.

Equation:

$$\begin{aligned} r &= \sqrt[3]{\tfrac{3V}{2\pi}} \\ &= \sqrt[3]{\tfrac{3 \cdot 100}{2 \cdot 3.14}} \\ &\approx \sqrt[3]{47.7707} \\ &\approx 3.63 \end{aligned}$$

Therefore, the radius is about 3.63 ft.

Determining the Domain of a Radical Function Composed with Other Functions

When radical functions are composed with other functions, determining domain can become more complicated.

Example:
Exercise:

Problem:

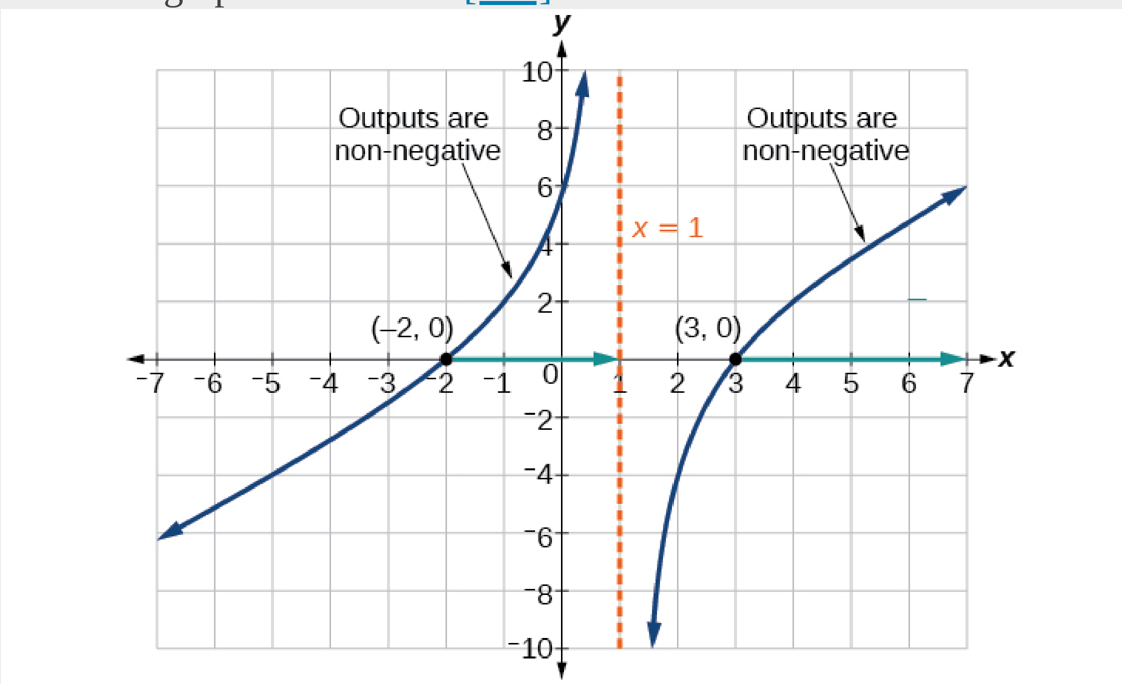
Finding the Domain of a Radical Function Composed with a Rational Function

Find the domain of the function $f(x)=\sqrt{\frac{(x+2)(x-3)}{(x-1)}}$.

Solution:

Because a square root is only defined when the quantity under the radical is non-negative, we need to determine where $\frac{(x+2)(x-3)}{(x-1)}\geq 0$. The output of a rational function can change signs (change from positive to negative or vice versa) at *x*-intercepts and at vertical asymptotes. For this equation, the graph could change signs at x = –2, 1, and 3.

To determine the intervals on which the rational expression is positive, we could test some values in the expression or sketch a graph. While both approaches work equally well, for this example we will use a graph as shown in [link].



This function has two *x*-intercepts, both of which exhibit linear behavior near the *x*-intercepts. There is one vertical asymptote,

corresponding to a linear factor; this behavior is similar to the basic reciprocal toolkit function, and there is no horizontal asymptote because the degree of the numerator is larger than the degree of the denominator. There is a *y*-intercept at $(0, \sqrt{6})$.

From the *y*-intercept and *x*-intercept at $x = -2,$ we can sketch the left side of the graph. From the behavior at the asymptote, we can sketch the right side of the graph.

From the graph, we can now tell on which intervals the outputs will be non-negative, so that we can be sure that the original function $f\left(x\right)$ will be defined. $f\left(x\right)$ has domain $-2 \le x < 1$ or $x \ge 3,$ or in interval notation, $\left[-2, 1\right) \cup \left[3, \infty\right).$

Finding Inverses of Rational Functions

As with finding inverses of quadratic functions, it is sometimes desirable to find the inverse of a rational function, particularly of rational functions that are the ratio of linear functions, such as in concentration applications.

Example:
Exercise:

Problem:
Finding the Inverse of a Rational Function

The function $C = \frac{20 + 0.4n}{100 + n}$ represents the concentration C of an acid solution after n mL of 40% solution has been added to 100 mL of a 20% solution. First, find the inverse of the function; that is, find an expression for n in terms of $C.$ Then use your result to determine how much of the 40% solution should be added so that the final mixture is a 35% solution.

Solution:

We first want the inverse of the function. We will solve for n in terms of C.

Equation:

$$\begin{aligned} C &= \frac{20+0.4n}{100+n} \\ C(100+n) &= 20+0.4n \\ 100C+Cn &= 20+0.4n \\ 100C-20 &= 0.4n-Cn \\ 100C-20 &= (0.4-C)n \\ n &= \frac{100C-20}{0.4-C} \end{aligned}$$

Now evaluate this function for $C=0.35\ (35\%)$.

Equation:

$$\begin{aligned} n &= \frac{100(0.35)-20}{0.4-0.35} \\ &= \frac{15}{0.05} \\ &= 300 \end{aligned}$$

We can conclude that 300 mL of the 40% solution should be added.

Note:
Exercise:

Problem: Find the inverse of the function $f(x)=\frac{x+3}{x-2}$.

Solution:

$f^{-1}(x)=\frac{2x+3}{x-1}$

Note:
Access these online resources for additional instruction and practice with inverses and radical functions.

- Graphing the Basic Square Root Function
- Find the Inverse of a Square Root Function
- Find the Inverse of a Rational Function
- Find the Inverse of a Rational Function and an Inverse Function Value
- Inverse Functions

Key Concepts

- The inverse of a quadratic function is a square root function.
- If f^{-1} is the inverse of a function f, then f is the inverse of the function f^{-1}. See [link].
- While it is not possible to find an inverse of most polynomial functions, some basic polynomials are invertible. See [link].
- To find the inverse of certain functions, we must restrict the function to a domain on which it will be one-to-one. See [link] and [link].
- When finding the inverse of a radical function, we need a restriction on the domain of the answer. See [link] and [link].
- Inverse and radical and functions can be used to solve application problems. See [link] and [link].

Section Exercises

Verbal

Exercise:

Problem:

Explain why we cannot find inverse functions for all polynomial functions.

Solution:

It can be too difficult or impossible to solve for x in terms of y.

Exercise:

Problem:

Why must we restrict the domain of a quadratic function when finding its inverse?

Exercise:

Problem:

When finding the inverse of a radical function, what restriction will we need to make?

Solution:

We will need a restriction on the domain of the answer.

Exercise:

Problem:

The inverse of a quadratic function will always take what form?

Algebraic

For the following exercises, find the inverse of the function on the given domain.

Exercise:

Problem: $f\left(x\right)={\left(x-4\right)}^{2},\text{ }\left[4,\infty \right)$

Solution:

$f^{-1}(x)=\sqrt{x}+4$

Exercise:

Problem: $f\left(x\right)=(x+2)^{2},\ [-2,\infty)$

Exercise:

Problem: $f(x)=(x+1)^{2}-3,\ [-1,\infty)$

Solution:

$f^{-1}(x)=\sqrt{x+3}-1$

Exercise:

Problem: $f(x)=2-\sqrt{3+x}$

Exercise:

Problem: $f(x)=3x^{2}+5,\ \ (-\infty,0]$

Solution:

$f^{-1}(x)=-\sqrt{\frac{x-5}{3}}$

Exercise:

Problem: $f\left(x\right)=12-x^{2},\ [0,\infty)$

Exercise:

Problem: $f\left(x\right)=9-x^{2},\ [0,\infty)$

Solution:

$f(x)=\sqrt{9-x}$

Exercise:

Problem: $f(x) = 2x^2 + 4,\ [0, \infty)$

For the following exercises, find the inverse of the functions.

Exercise:

Problem: $f(x) = x^3 + 5$

Solution:

$f^{-1}(x) = \sqrt[3]{x - 5}$

Exercise:

Problem: $f\left(x\right) = 3x^3 + 1$

Exercise:

Problem: $f(x) = 4 - x^3$

Solution:

$f^{-1}(x) = \sqrt[3]{4 - x}$

Exercise:

Problem: $f\left(x\right) = 4 - 2x^3$

For the following exercises, find the inverse of the functions.

Exercise:

Problem: $f(x) = \sqrt{2x + 1}$

Solution:

$f^{-1}(x) = \frac{x^2 - 1}{2},\ \ [0, \infty)$

Exercise:

Problem: $f(x) = \sqrt{3 - 4x}$

Exercise:

Problem: $f\left(x\right) = 9 + \sqrt{4x - 4}$

Solution:

$f^{-1}(x) = \frac{(x-9)^2+4}{4}, \quad [9, \infty)$

Exercise:

Problem: $f\left(x\right) = \sqrt{6x - 8} + 5$

Exercise:

Problem: $f\left(x\right) = 9 + 2\sqrt[3]{x}$

Solution:

$f^{-1}(x) = \left(\frac{x-9}{2}\right)^3$

Exercise:

Problem: $f\left(x\right) = 3 - \sqrt[3]{x}$

Exercise:

Problem: $f\left(x\right) = \frac{2}{x+8}$

Solution:

$f^{-1}(x) = \frac{2-8x}{x}$

Exercise:

Problem: $f\left(x\right) = \frac{3}{x-4}$

Exercise:

Problem: $f\left(x\right) = \frac{x+3}{x+7}$

Solution:

$f^{-1}(x) = \frac{7x-3}{1-x}$

Exercise:

Problem: $f\left(x\right) = \frac{x-2}{x+7}$

Exercise:

Problem: $f\left(x\right) = \frac{3x+4}{5-4x}$

Solution:

$f^{-1}(x) = \frac{5x-4}{4x+3}$

Exercise:

Problem: $f\left(x\right) = \frac{5x+1}{2-5x}$

Exercise:

Problem: $f(x) = x^2 + 2x,\ [-1, \infty)$

Solution:

$f^{-1}(x) = \sqrt{x+1} - 1$

Exercise:

Problem: $f(x) = x^2 + 4x + 1,\ [-2, \infty)$

Exercise:

Problem: $f(x) = x^2 - 6x + 3,\ [3, \infty)$

Solution:

$f^{-1}(x) = \sqrt{x+6} + 3$

Graphical

For the following exercises, find the inverse of the function and graph both the function and its inverse.

Exercise:

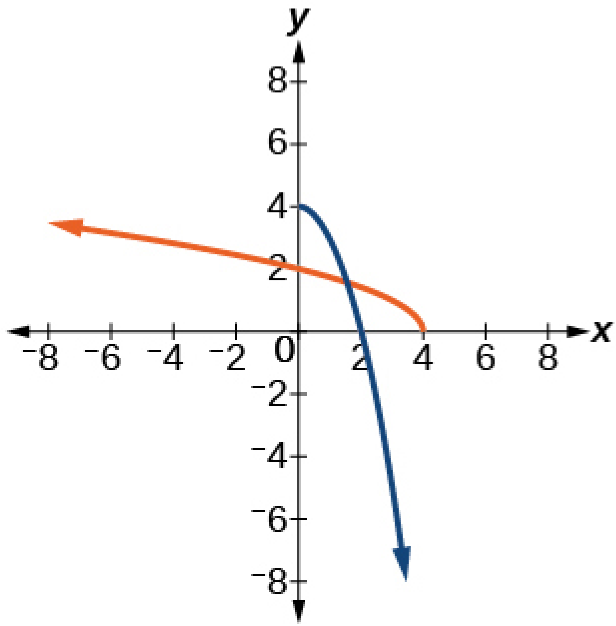
Problem: $f(x) = x^2 + 2,\ x \geq 0$

Exercise:

Problem: $f(x) = 4 - x^2,\ x \geq 0$

Solution:

$f^{-1}(x) = \sqrt{4 - x}$



Exercise:

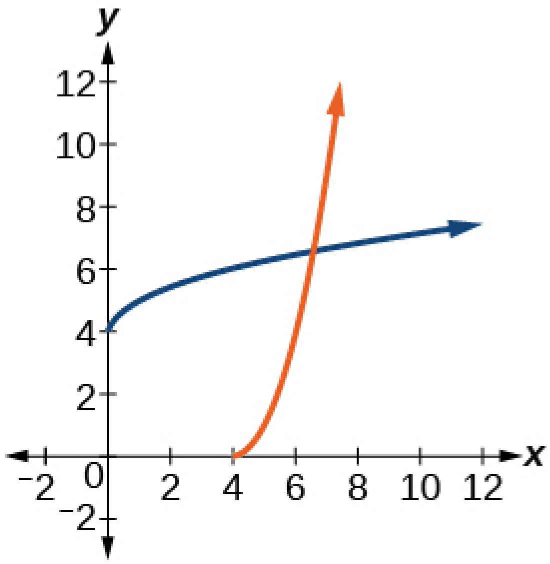
Problem: $f(x) = (x+3)^2,\ x \geq -3$

Exercise:

Problem: $f(x) = (x-4)^2,\ x \geq 4$

Solution:

$f^{-1}(x) = \sqrt{x} + 4$



Exercise:

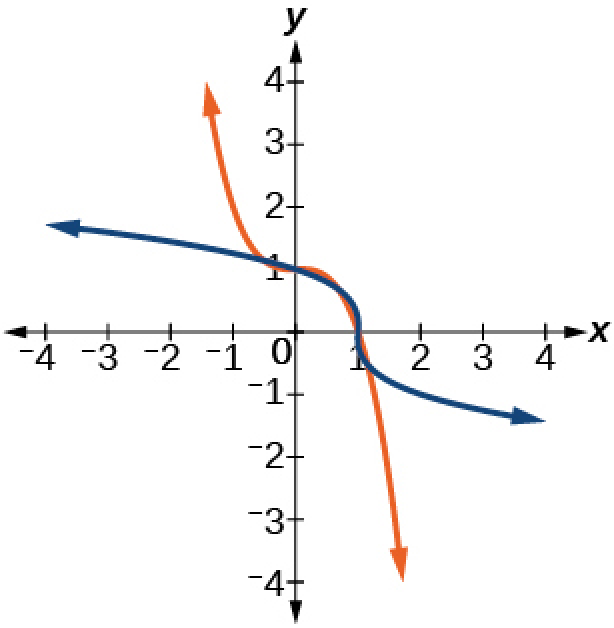
Problem: $f(x) = x^3 + 3$

Exercise:

Problem: $f(x) = 1 - x^3$

Solution:

$f^{-1}(x) = \sqrt[3]{1 - x}$



Exercise:

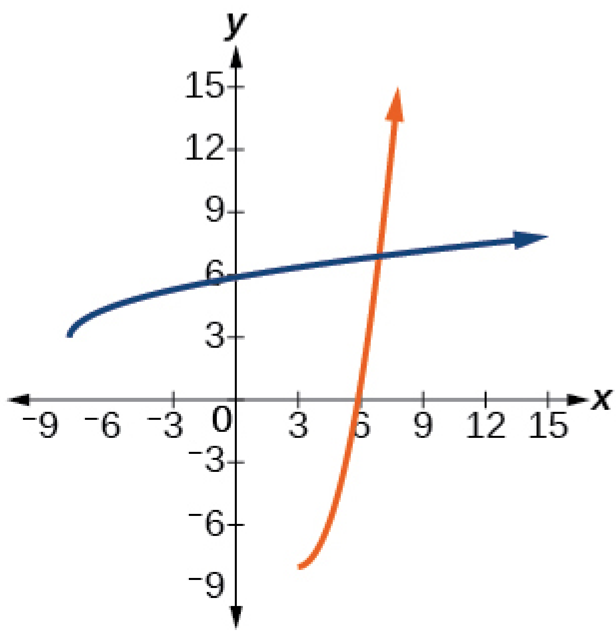
Problem: $f(x) = x^2 + 4x,\ x \geq -2$

Exercise:

Problem: $f(x) = x^2 - 6x + 1,\ x \geq 3$

Solution:

$f^{-1}(x) = \sqrt{x+8} + 3$



Exercise:

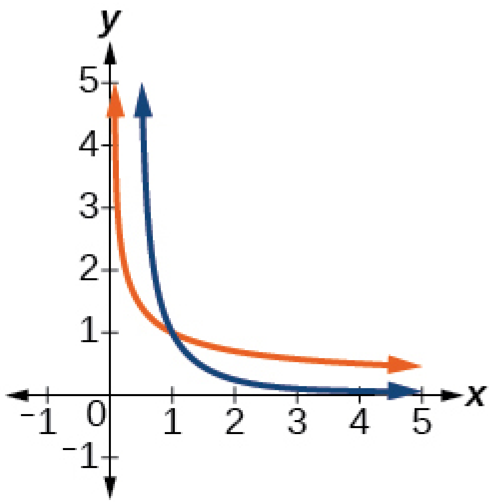
Problem: $f(x) = \frac{2}{x}$

Exercise:

Problem: $f(x) = \frac{1}{x^2}, \ x \geq 0$

Solution:

$f^{-1}(x) = \sqrt{\frac{1}{x}}$



For the following exercises, use a graph to help determine the domain of the functions.

Exercise:

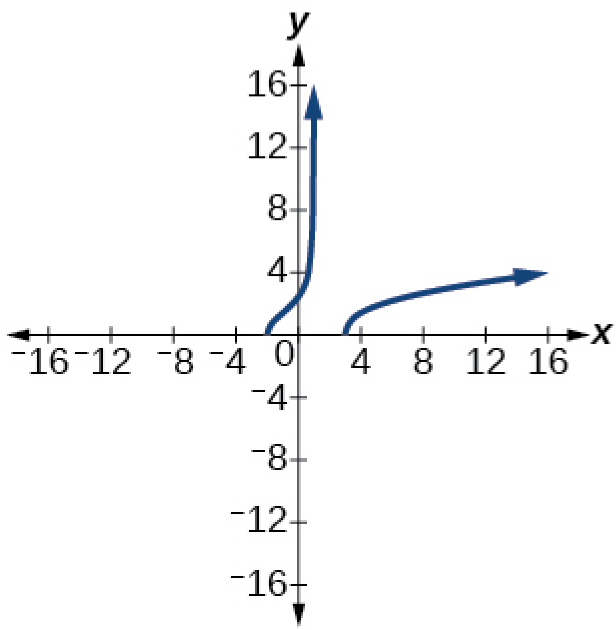
Problem: $f(x) = \sqrt{\frac{(x+1)(x-1)}{x}}$

Exercise:

Problem: $f(x) = \sqrt{\frac{(x+2)(x-3)}{x-1}}$

Solution:

$[-2, 1) \cup [3, \infty)$



Exercise:

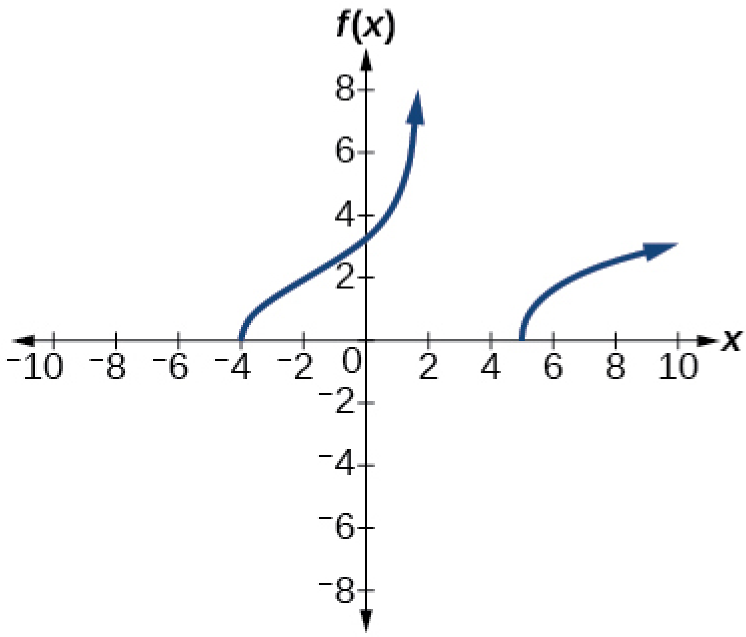
Problem: $f(x) = \sqrt{\frac{x(x+3)}{x-4}}$

Exercise:

Problem: $f(x) = \sqrt{\frac{x^2-x-20}{x-2}}$

Solution:

$[-4, 2) \cup [5, \infty)$



Exercise:

Problem: $f(x) = \sqrt{\frac{9-x^2}{x+4}}$

Technology

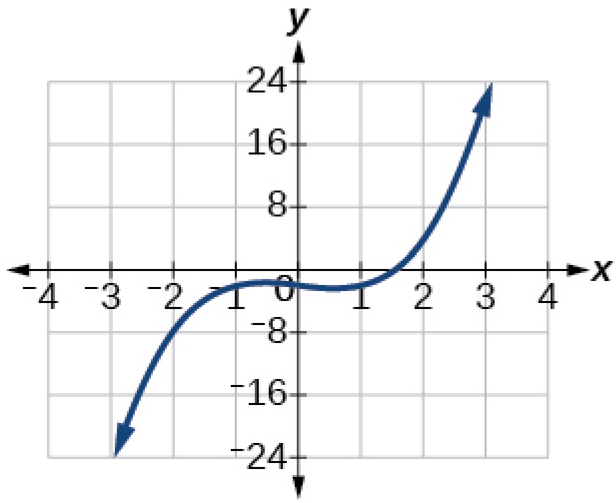
For the following exercises, use a calculator to graph the function. Then, using the graph, give three points on the graph of the inverse with *y*-coordinates given.

Exercise:

Problem: $f(x) = x^3 - x - 2, \, y = 1, \; 2, \; 3$

Solution:

$(-2, \; 0); \; (4, \; 2); \; (22, \; 3)$



Exercise:

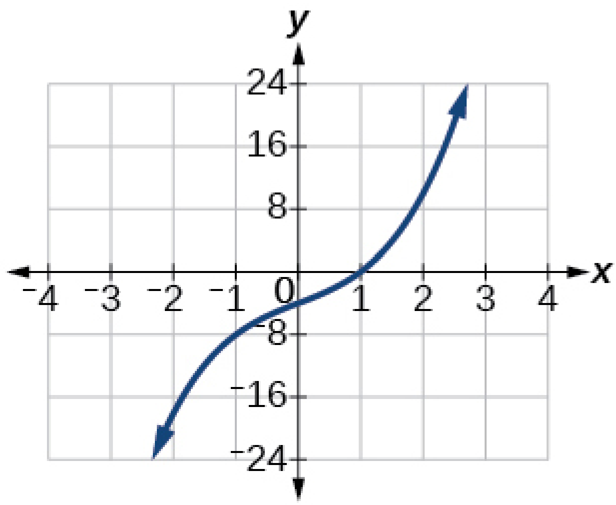
Problem: $f(x) = x^3 + x - 2,\ y = 0,\ 1,\ 2$

Exercise:

Problem: $f(x) = x^3 + 3x - 4,\ y = 0,\ 1,\ 2$

Solution:

$(-4,\ 0);\ (0,\ 1);\ (10,\ 2)$



Exercise:

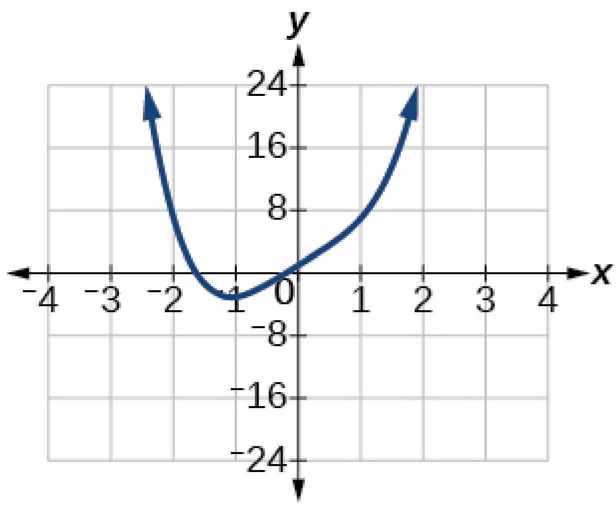
Problem: $f(x) = x^3 + 8x - 4,\ y = -1,\ 0,\ 1$

Exercise:

Problem: $f(x) = x^4 + 5x + 1,\ y = -1,\ 0,\ 1$

Solution:

$(-3,\ -1);\ (1,\ 0);\ (7,\ 1)$



Extensions

For the following exercises, find the inverse of the functions with a, b, c positive real numbers.

Exercise:

Problem: $f(x) = ax^3 + b$

Exercise:

Problem: $f(x) = x^2 + bx$

Solution:

$f^{-1}(x) = \sqrt{x + \frac{b^2}{4}} - \frac{b}{2}$

Exercise:

Problem: $f(x) = \sqrt{ax^2 + b}$

Exercise:

Problem: $f(x) = \sqrt[3]{ax + b}$

Solution:

$f^{-1}(x) = \frac{x^3 - b}{a}$

Exercise:

Problem: $f(x) = \frac{ax+b}{x+c}$

Real-World Applications

For the following exercises, determine the function described and then use it to answer the question.

Exercise:

Problem:

An object dropped from a height of 200 meters has a height, $h\left(t\right),$ in meters after t seconds have lapsed, such that $h(t) = 200 - 4.9t^2$. Express t as a function of height, $h,$ and find the time to reach a height of 50 meters.

Solution:

$t(h) = \sqrt{\frac{200-h}{4.9}}$, 5.53 seconds

Exercise:

Problem:

An object dropped from a height of 600 feet has a height, $h\,(t),$ in feet after t seconds have elapsed, such that $h(t) = 600 - 16t^2.$ Express t as a function of height $h,$ and find the time to reach a height of 400 feet.

Exercise:

Problem:

The volume, $V,$ of a sphere in terms of its radius, $r,$ is given by $V(r) = \frac{4}{3}\pi r^3.$ Express r as a function of $V,$ and find the radius of a sphere with volume of 200 cubic feet.

Solution:

$r(V) = \sqrt[3]{\frac{3V}{4\pi}}$, 3.63 feet

Exercise:

Problem:

The surface area, $A,$ of a sphere in terms of its radius, $r,$ is given by $A(r) = 4\pi r^2.$ Express r as a function of $V,$ and find the radius of a sphere with a surface area of 1000 square inches.

Exercise:

Problem:

A container holds 100 ml of a solution that is 25 ml acid. If n ml of a solution that is 60% acid is added, the function $C(n) = \frac{25+.6n}{100+n}$ gives the concentration, $C,$ as a function of the number of ml added, $n.$ Express n as a function of C and determine the number of mL that need to be added to have a solution that is 50% acid.

Solution:

$n(C) = \frac{100C-25}{.6-C}$, 250 mL

Exercise:

Problem:

The period $T,$ in seconds, of a simple pendulum as a function of its length $l,$ in feet, is given by $T(l) = 2\pi\sqrt{\frac{l}{32.2}}$. Express l as a function of T and determine the length of a pendulum with period of 2 seconds.

Exercise:

Problem:

The volume of a cylinder , $V,$ in terms of radius, $r,$ and height, $h,$ is given by $V = \pi r^2 h.$ If a cylinder has a height of 6 meters, express the radius as a function of V and find the radius of a cylinder with volume of 300 cubic meters.

Solution:

$r(V) = \sqrt{\frac{V}{6\pi}}$, 3.99 meters

Exercise:

Problem:

The surface area, $A,$ of a cylinder in terms of its radius, $r,$ and height, $h,$ is given by $A=2\pi r^2+2\pi rh.$ If the height of the cylinder is 4 feet, express the radius as a function of V and find the radius if the surface area is 200 square feet.

Exercise:

Problem:

The volume of a right circular cone, $V,$ in terms of its radius, $r,$ and its height, $h,$ is given by $V=\frac{1}{3}\pi r^2h.$ Express r in terms of V if the height of the cone is 12 feet and find the radius of a cone with volume of 50 cubic inches.

Solution:

$r(V)=\sqrt{\frac{V}{4\pi}},$ 1.99 inches

Exercise:

Problem:

Consider a cone with height of 30 feet. Express the radius, $r,$ in terms of the volume, $V,$ and find the radius of a cone with volume of 1000 cubic feet.

Glossary

invertible function
: any function that has an inverse function

Dividing Polynomials
In this section, you will:

- Use long division to divide polynomials.
- Use synthetic division to divide polynomials.

Lincoln Memorial, Washington, D.C. (credit: Ron Cogswell, Flickr)

The exterior of the Lincoln Memorial in Washington, D.C., is a large rectangular solid with length 61.5 meters (m), width 40 m, and height 30 m. [footnote] We can easily find the volume using elementary geometry.
National Park Service. "Lincoln Memorial Building Statistics." http://www.nps.gov/linc/historyculture/lincoln-memorial-building-statistics.htm. Accessed 4/3/2014
Equation:

$$\begin{aligned} V &= l \cdot w \cdot h \\ &= 61.5 \cdot 40 \cdot 30 \\ &= 73{,}800 \end{aligned}$$

So the volume is 73,800 cubic meters (m^3). Suppose we knew the volume, length, and width. We could divide to find the height.

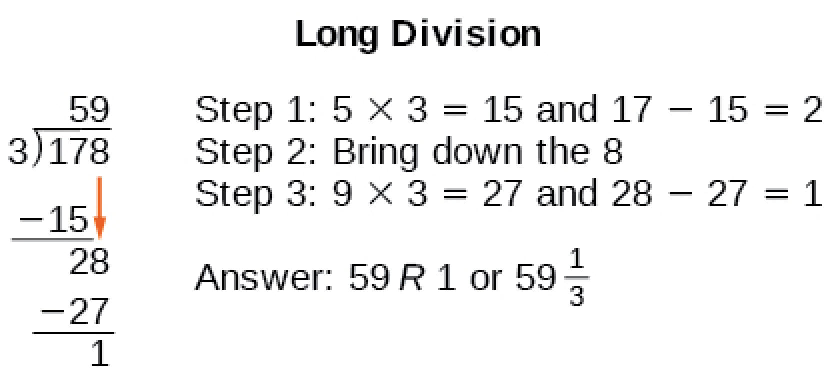
Equation:

$$\begin{aligned} h &= \frac{V}{l \cdot w} \\ &= \frac{73{,}800}{61.5 \cdot 40} \\ &= 30 \end{aligned}$$

As we can confirm from the dimensions above, the height is 30 m. We can use similar methods to find any of the missing dimensions. We can also use the same method if any or all of the measurements contain variable expressions. For example, suppose the volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$; the width is given by $x - 2$. To find the height of the solid, we can use polynomial division, which is the focus of this section.

Using Long Division to Divide Polynomials

We are familiar with the long division algorithm for ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let’s divide 178 by 3 using long division.



Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

Equation:

$$\begin{aligned} \text{dividend} &= (\text{divisor} \ \cdot \ \text{quotient}) + \text{remainder} \\ 178 &= (3 \cdot 59) + 1 \\ &= 177 + 1 \\ &= 178 \end{aligned}$$

We call this the **Division Algorithm** and will discuss it more formally after looking at an example.

Division of polynomials that contain more than one term has similarities to long division of whole numbers. We can write a polynomial dividend as the product of the divisor and the quotient added to the remainder. The terms of the polynomial division correspond to the digits (and place values) of the whole number division. This method allows us to divide two polynomials. For example, if we were to divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm, it would look like this:

$x + 2 \overline{\smash{)}\,2x^3 - 3x^2 + 4x + 5}$	Set up the division problem.
$\begin{array}{r} 2x^2 \\ x + 2 \overline{\smash{)}\,2x^3 - 3x^2 + 4x + 5} \end{array}$	$2x^3$ divided by x is $2x^2$.
$\begin{array}{r} 2x^2 \\ x + 2 \overline{\smash{)}\,2x^3 - 3x^2 + 4x + 5} \\ \underline{-(2x^3 + 4x^2)} \\ -7x^2 + 4x \end{array}$	Multiply $x + 2$ by $2x^2$. Subtract.
$\begin{array}{r} 2x^2 - 7x \\ x + 2 \overline{\smash{)}\,2x^3 - 3x^2 + 4x + 5} \\ \underline{-(2x^3 + 4x^2)} \\ -7x^2 + 4x \\ \underline{-(-7x^2 + 14x)} \\ 18x + 5 \end{array}$	Bring down the next term. $-7x^2$ divided by x is $-7x$. Multiply $x + 2$ by $-7x$. Subtract. Bring down the next term.
$\begin{array}{r} 2x^2 - 7x + 18 \\ x + 2 \overline{\smash{)}\,2x^3 - 3x^2 + 4x + 5} \\ \underline{-(2x^3 + 4x^2)} \\ -7x^2 + 4x \\ \underline{-(-7x^2 + 14x)} \\ 18x + 5 \\ \underline{-18x + 36} \\ -31 \end{array}$	$18x$ divided by x is 18. Multiply $x + 2$ by 18. Subtract.

We have found

Equation:

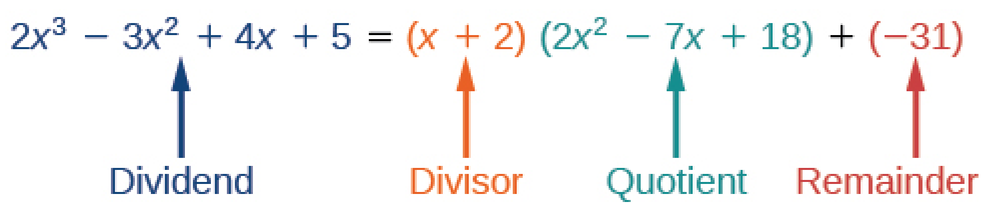
$$\frac{2x^3 - 3x^2 + 4x + 5}{x+2} = 2x^2 - 7x + 18 - \frac{31}{x+2}$$

or

Equation:

$$\frac{2x^3 - 3x^2 + 4x + 5}{x+2} = (x+2)(2x^2 - 7x + 18) - 31$$

We can identify the dividend, the divisor, the quotient, and the remainder.



Writing the result in this manner illustrates the Division Algorithm.

Note:
The Division Algorithm
The **Division Algorithm** states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

Equation:

$$f(x) = d(x)q(x) + r(x)$$

$q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.
If $r(x) = 0,$ then $d(x)$ divides evenly into $f(x).$ This means that, in this case, both $d(x)$ and $q(x)$ are factors of $f(x).$

Note: Given a polynomial and a binomial, use long division to divide the polynomial by the binomial.

1. Set up the division problem.
2. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
3. Multiply the answer by the divisor and write it below the like terms of the dividend.
4. Subtract the bottom binomial from the top binomial.
5. Bring down the next term of the dividend.
6. Repeat steps 2–5 until reaching the last term of the dividend.
7. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

Example:
Exercise:

Problem:
Using Long Division to Divide a Second-Degree Polynomial

Divide $5x^2 + 3x - 2$ by $x + 1$.

Solution:

$x+1\ \overline{)\,5x^2+3x-2}$	Set up division problem.
$\begin{array}{r} 5x \\ x+1\ \overline{)\,5x^2+3x-2} \end{array}$	$5x^2$ divided by x is $5x$.
$\begin{array}{r} 5x \\ x+1\ \overline{)\,5x^2+3x-2} \\ \underline{-(5x^2+5x)} \\ -2x-2 \end{array}$	Multiply $x+1$ by $5x$. Subtract.
$\begin{array}{r} 5x-2 \\ x+1\ \overline{)\,5x^2+3x-2} \\ \underline{-(5x^2+5x)} \\ -2x-2 \\ \underline{-(-2x-2)} \\ 0 \end{array}$	Bring down the next term. $-2x$ divided by x is -2. Multiply $x+1$ by -2. Subtract.

The quotient is $5x - 2$. The remainder is 0. We write the result as

Equation:

$$\frac{5x^2 + 3x - 2}{x + 1} = 5x - 2$$

or

Equation:

$$5x^2 + 3x - 2 = (x + 1)(5x - 2)$$

Analysis

This division problem had a remainder of 0. This tells us that the dividend is divided evenly by the divisor, and that the divisor is a factor of the dividend.

Example:

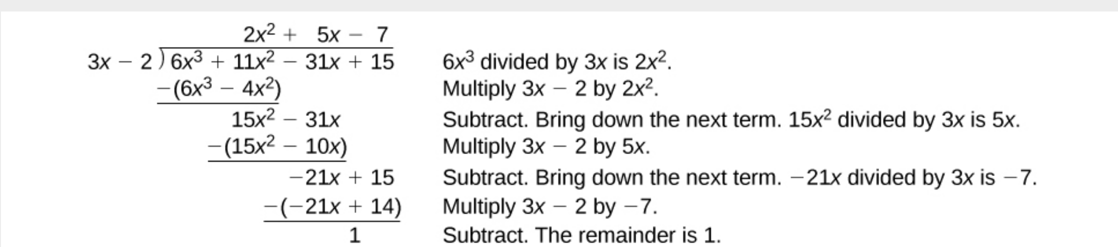
Exercise:

Problem:

Using Long Division to Divide a Third-Degree Polynomial

Divide $6x^3 + 11x^2 - 31x + 15$ by $3x - 2$.

Solution:



There is a remainder of 1. We can express the result as:

Equation:

$$\frac{6x^3 + 11x^2 - 31x + 15}{3x - 2} = 2x^2 + 5x - 7 + \frac{1}{3x - 2}$$

Analysis

We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.
Equation:

$$(3x - 2)(2x^2 + 5x - 7) + 1 = 6x^3 + 11x^2 - 31x + 15$$

Notice, as we write our result,

- the dividend is $6x^3 + 11x^2 - 31x + 15$
- the divisor is $3x - 2$
- the quotient is $2x^2 + 5x - 7$
- the remainder is 1

Note:
Exercise:

Problem: Divide $16x^3 - 12x^2 + 20x - 3$ by $4x + 5$.

Solution:

$$4x^2 - 8x + 15 - \frac{78}{4x + 5}$$

Using Synthetic Division to Divide Polynomials

As we’ve seen, long division of polynomials can involve many steps and be quite cumbersome. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1.

To illustrate the process, recall the example at the beginning of the section.

Divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm.

The final form of the process looked like this:

$$
\begin{array}{r l}
 & \quad 2x^2 + \quad x + 18 \\
x+2 & \overline{)\,2x^3 - 3x^2 + 4x + 5} \\
 & \underline{-(2x^3 + 4x^2)} \\
 & \qquad -7x^2 + 4x \\
 & \quad \underline{-(-7x^2 - 14x)} \\
 & \qquad\qquad 18x + 5 \\
 & \qquad\quad \underline{-(18x + 36)} \\
 & \qquad\qquad\quad -31
\end{array}
$$

There is a lot of repetition in the table. If we don’t write the variables but, instead, line up their coefficients in columns under the division sign and also eliminate the partial products, we already have a simpler version of the entire problem.

$$
\begin{array}{r|rrrr}
2 & 2 & -3 & 4 & 5 \\
 & \underline{-2} & \underline{-4} & & \\
 & & \underline{-7} & \underline{14} & \\
 & & & \underline{18} & \underline{-36} \\
 & & & & -31
\end{array}
$$

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of

the “divisor” to –2, multiply and add. The process starts by bringing down the leading coefficient.

$$\begin{array}{r|rrrr} -2 & 2 & -3 & 4 & 5 \\ & & -4 & 14 & -36 \\ \hline & 2 & -7 & 18 & -31 \end{array}$$

We then multiply it by the “divisor” and add, repeating this process column by column, until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is $2x^2 – 7x + 18$ and the remainder is $–31.$ The process will be made more clear in [link].

Note:
Synthetic Division
Synthetic division is a shortcut that can be used when the divisor is a binomial in the form $x - k.$ In **synthetic division**, only the coefficients are used in the division process.

Note:
Given two polynomials, use synthetic division to divide.

1. Write k for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by $k.$ Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by $k.$ Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, and so on.

Example:
Exercise:

Problem:
Using Synthetic Division to Divide a Second-Degree Polynomial

Use synthetic division to divide $5x^2 - 3x - 36$ by $x - 3$.

Solution:

Begin by setting up the synthetic division. Write k and the coefficients.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & & \\ \hline & & & \end{array}$$

Bring down the lead coefficient. Multiply the lead coefficient by k.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & \\ \hline & 5 & & \end{array}$$

Continue by adding the numbers in the second column. Multiply the resulting number by k. Write the result in the next column. Then add the numbers in the third column.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & 36 \\ \hline & 5 & 12 & 0 \end{array}$$

The result is $5x + 12$. The remainder is 0. So $x - 3$ is a factor of the original polynomial.

Analysis

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.

$$(x-3)(5x+12)+0=5x^2-3x-36$$

Example:
Exercise:

Problem:
Using Synthetic Division to Divide a Third-Degree Polynomial

Use synthetic division to divide $4x^3+10x^2-6x-20$ by $x+2.$

Solution:

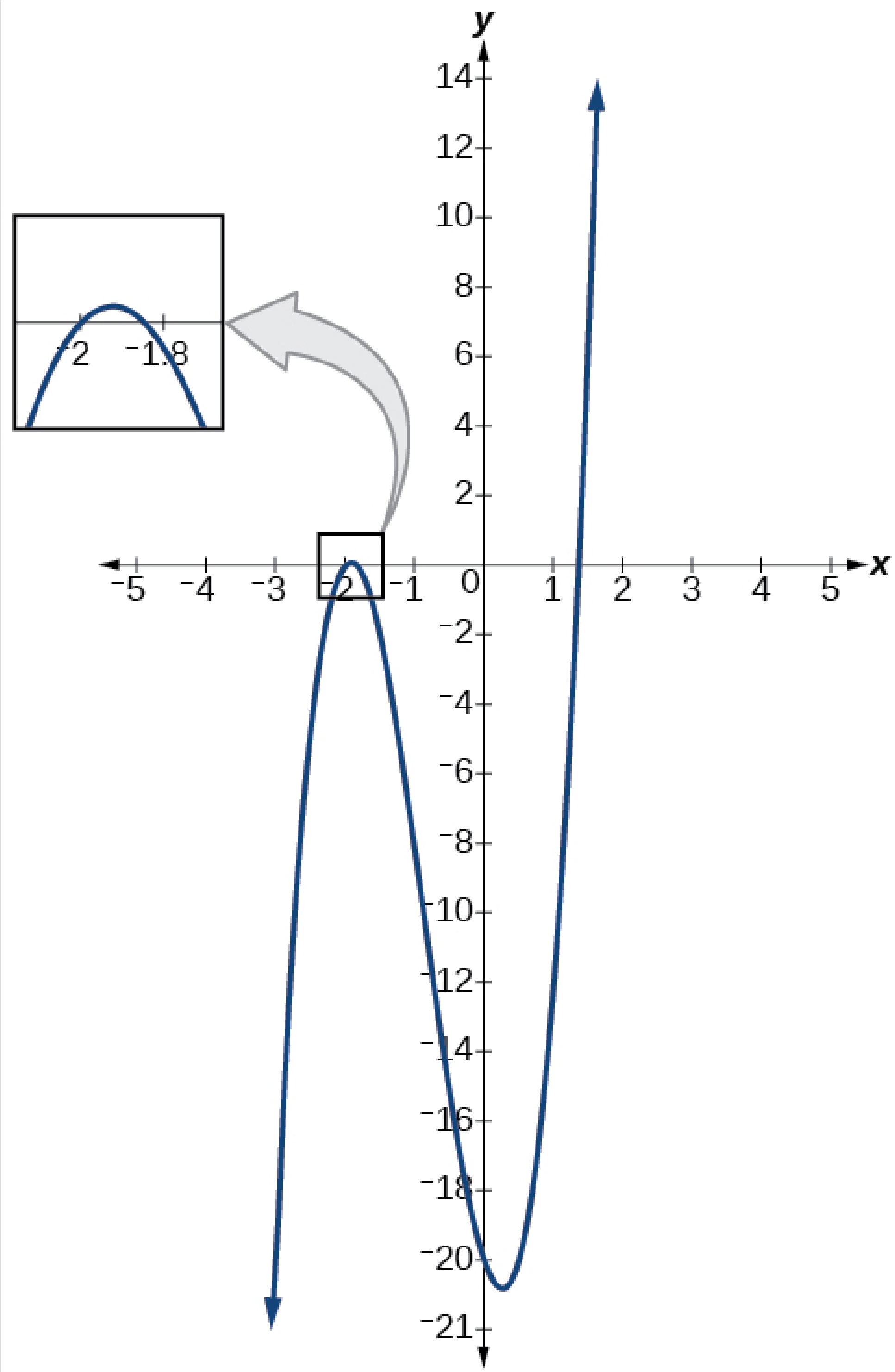
The binomial divisor is $x+2$ so $k=-2.$ Add each column, multiply the result by –2, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -2 & 4 & 10 & -6 & -20 \\ & & -8 & -4 & 20 \\ \hline & 4 & 2 & -10 & 0 \end{array}$$

The result is $4x^2+2x-10.$ The remainder is 0. Thus, $x+2$ is a factor of $4x^3+10x^2-6x-20.$

Analysis

The graph of the polynomial function $f(x)=4x^3+10x^2-6x-20$ in [link] shows a zero at $x=k=-2.$ This confirms that $x+2$ is a factor of $4x^3+10x^2-6x-20.$



Example:
Exercise:

Problem:
Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9x^4+10x^3+7x^2-6$ by $x-1$.

Solution:

Notice there is no *x*-term. We will use a zero as the coefficient for that term.

$$
\begin{array}{r|rrrrr}
1 & -9 & 10 & 7 & 0 & -6 \\
 & & -9 & 1 & 8 & 8 \\
\hline
 & -9 & 1 & 8 & 8 & 2
\end{array}
$$

The result is $-9x^3+x^2+8x+8+\frac{2}{x-1}$.

Note:
Exercise:

Problem:

Use synthetic division to divide $3x^4+18x^3-3x+40$ by $x+7$.

Solution:

$3x^3-3x^2+21x-150+\frac{1{,}090}{x+7}$

Using Polynomial Division to Solve Application Problems

Polynomial division can be used to solve a variety of application problems involving expressions for area and volume. We looked at an application at

the beginning of this section. Now we will solve that problem in the following example.

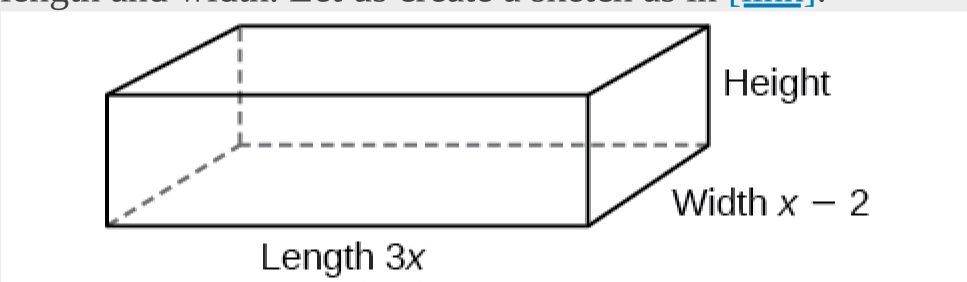
Example:
Exercise:

Problem:
Using Polynomial Division in an Application Problem

The volume of a rectangular solid is given by the polynomial $3x^4-3x^3-33x^2+54x.$ The length of the solid is given by $3x$ and the width is given by $x-2.$ Find the height of the solid.

Solution:

There are a few ways to approach this problem. We need to divide the expression for the volume of the solid by the expressions for the length and width. Let us create a sketch as in [link].



We can now write an equation by substituting the known values into the formula for the volume of a rectangular solid.

Equation:

$$V=l\cdot w\cdot h$$
$$3x^4-3x^3-33x^2+54x=3x\cdot(x-2)\cdot h$$

To solve for $h,$ first divide both sides by $3x.$

Equation:

$$\frac{3x\cdot(x-2)\cdot h}{3x} = \frac{3x^4-3x^3-33x^2+54x}{3x}$$
$$(x-2)h = x^3 - x^2 - 11x + 18$$

Now solve for h using synthetic division.

Equation:

$$h = \frac{x^3 - x^2 - 11x + 18}{x - 2}$$

Equation:

$$\begin{array}{r|rrrr} 2 & 1 & -1 & -11 & 18 \\ & & 2 & 2 & -18 \\ \hline & 1 & 1 & -9 & 0 \end{array}$$

The quotient is $x^2 + x - 9$ and the remainder is 0. The height of the solid is $x^2 + x - 9$.

Note:
Exercise:

Problem:

The area of a rectangle is given by $3x^3 + 14x^2 - 23x + 6.$ The width of the rectangle is given by $x + 6.$ Find an expression for the length of the rectangle.

Solution:

$3x^2 - 4x + 1$

Note:
Access these online resources for additional instruction and practice with polynomial division.

- Dividing a Trinomial by a Binomial Using Long Division
- Dividing a Polynomial by a Binomial Using Long Division
- Ex 2: Dividing a Polynomial by a Binomial Using Synthetic Division
- Ex 4: Dividing a Polynomial by a Binomial Using Synthetic Division

Key Equations

Division Algorithm	$f(x) = d(x)q(x) + r(x)$ where $q(x) \neq 0$

Key Concepts

- Polynomial long division can be used to divide a polynomial by any polynomial with equal or lower degree. See [link] and [link].
- The Division Algorithm tells us that a polynomial dividend can be written as the product of the divisor and the quotient added to the remainder.
- Synthetic division is a shortcut that can be used to divide a polynomial by a binomial in the form $x - k$. See [link], [link], and [link].
- Polynomial division can be used to solve application problems, including area and volume. See [link].

Section Exercises

Verbal

Exercise:

Problem:

If division of a polynomial by a binomial results in a remainder of zero, what can be conclude?

Solution:

The binomial is a factor of the polynomial.

Exercise:

Problem:

If a polynomial of degree n is divided by a binomial of degree 1, what is the degree of the quotient?

Algebraic

For the following exercises, use long division to divide. Specify the quotient and the remainder.

Exercise:

Problem: $\left(x^2 + 5x - 1\right) \div (x - 1)$

Solution:

$x + 6 + \frac{5}{x-1}$, quotient: $x + 6$, remainder: 5

Exercise:

Problem: $\left(2x^2 - 9x - 5\right) \div (x - 5)$

Exercise:

Problem: $\left(3x^2 + 23x + 14\right) \div (x + 7)$

Solution:

$3x + 2$, quotient: $3x + 2$, remainder: 0

Exercise:

Problem: $\left(4x^2 - 10x + 6\right) \div (4x + 2)$

Exercise:

Problem: $\left(6x^2 - 25x - 25\right) \div (6x + 5)$

Solution:

$x - 5$, quotient: $x - 5$, remainder: 0

Exercise:

Problem: $\left(-x^2 - 1\right) \div (x + 1)$

Exercise:

Problem: $\left(2x^2 - 3x + 2\right) \div (x + 2)$

Solution:

$2x - 7 + \frac{16}{x+2}$, quotient: $2x - 7$, remainder: 16

Exercise:

Problem: $\left(x^3 - 126\right) \div (x - 5)$

Exercise:

Problem: $\left(3x^2 - 5x + 4\right) \div (3x + 1)$

Solution:

$x-2+\frac{6}{3x+1}$, quotient: $x-2$, remainder: 6

Exercise:

Problem: $\left(x^3-3x^2+5x-6\right)\div(x-2)$

Exercise:

Problem: $\left(2x^3+3x^2-4x+15\right)\div(x+3)$

Solution:

$2x^2-3x+5$, quotient: $2x^2-3x+5$, remainder: 0

For the following exercises, use synthetic division to find the quotient.

Exercise:

Problem: $\left(3x^3-2x^2+x-4\right)\div(x+3)$

Exercise:

Problem: $\left(2x^3-6x^2-7x+6\right)\div(x-4)$

Solution:

$2x^2+2x+1+\frac{10}{x-4}$

Exercise:

Problem: $\left(6x^3-10x^2-7x-15\right)\div(x+1)$

Exercise:

Problem: $\left(4x^3-12x^2-5x-1\right)\div(2x+1)$

Solution:

$2x^2 - 7x + 1 - \frac{2}{2x+1}$

Exercise:

Problem: $\left(9x^3 - 9x^2 + 18x + 5\right) \div (3x - 1)$

Exercise:

Problem: $\left(3x^3 - 2x^2 + x - 4\right) \div (x + 3)$

Solution:

$3x^2 - 11x + 34 - \frac{106}{x+3}$

Exercise:

Problem: $\left(-6x^3 + x^2 - 4\right) \div (2x - 3)$

Exercise:

Problem: $\left(2x^3 + 7x^2 - 13x - 3\right) \div (2x - 3)$

Solution:

$x^2 + 5x + 1$

Exercise:

Problem: $\left(3x^3 - 5x^2 + 2x + 3\right) \div (x + 2)$

Exercise:

Problem: $\left(4x^3 - 5x^2 + 13\right) \div (x + 4)$

Solution:

$4x^2 - 21x + 84 - \frac{323}{x+4}$

Exercise:

Problem: $\left(x^3 - 3x + 2\right) \div (x + 2)$

Exercise:

Problem: $\left(x^3 - 21x^2 + 147x - 343\right) \div (x - 7)$

Solution:

$x^2 - 14x + 49$

Exercise:

Problem: $\left(x^3 - 15x^2 + 75x - 125\right) \div (x - 5)$

Exercise:

Problem: $\left(9x^3 - x + 2\right) \div (3x - 1)$

Solution:

$3x^2 + x + \frac{2}{3x-1}$

Exercise:

Problem: $\left(6x^3 - x^2 + 5x + 2\right) \div (3x + 1)$

Exercise:

Problem: $\left(x^4 + x^3 - 3x^2 - 2x + 1\right) \div (x + 1)$

Solution:

$x^3 - 3x + 1$

Exercise:

Problem: $\left(x^4 - 3x^2 + 1\right) \div (x - 1)$

Exercise:

Problem: $\left(x^4 + 2x^3 - 3x^2 + 2x + 6\right) \div (x + 3)$

Solution:

$x^3 - x^2 + 2$

Exercise:

Problem: $\left(x^4 - 10x^3 + 37x^2 - 60x + 36\right) \div (x - 2)$

Exercise:

Problem: $\left(x^4 - 8x^3 + 24x^2 - 32x + 16\right) \div (x - 2)$

Solution:

$x^3 - 6x^2 + 12x - 8$

Exercise:

Problem: $\left(x^4 + 5x^3 - 3x^2 - 13x + 10\right) \div (x + 5)$

Exercise:

Problem: $\left(x^4 - 12x^3 + 54x^2 - 108x + 81\right) \div (x - 3)$

Solution:

$x^3 - 9x^2 + 27x - 27$

Exercise:

Problem: $\left(4x^4 - 2x^3 - 4x + 2\right) \div (2x - 1)$

Exercise:

Problem: $\left(4x^4 + 2x^3 - 4x^2 + 2x + 2\right) \div (2x + 1)$

Solution:

$2x^3 - 2x + 2$

For the following exercises, use synthetic division to determine whether the first expression is a factor of the second. If it is, indicate the factorization.

Exercise:

Problem: $x - 2,\ 4x^3 - 3x^2 - 8x + 4$

Exercise:

Problem: $x - 2,\ 3x^4 - 6x^3 - 5x + 10$

Solution:

Yes $(x - 2)(3x^3 - 5)$

Exercise:

Problem: $x + 3,\ -4x^3 + 5x^2 + 8$

Exercise:

Problem: $x - 2,\ 4x^4 - 15x^2 - 4$

Solution:

Yes $(x - 2)(4x^3 + 8x^2 + x + 2)$

Exercise:

Problem:$x-\frac{1}{2}$, $2x^4-x^3+2x-1$

Exercise:

Problem:$x+\frac{1}{3}$, $3x^4+x^3-3x+1$

Solution:

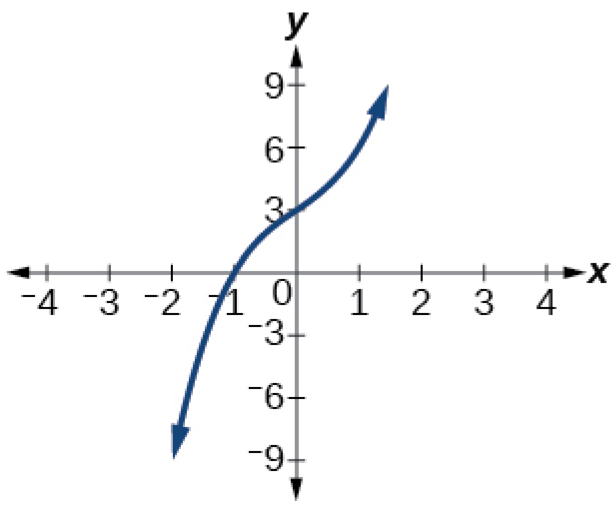
No

Graphical

For the following exercises, use the graph of the third-degree polynomial and one factor to write the factored form of the polynomial suggested by the graph. The leading coefficient is one.

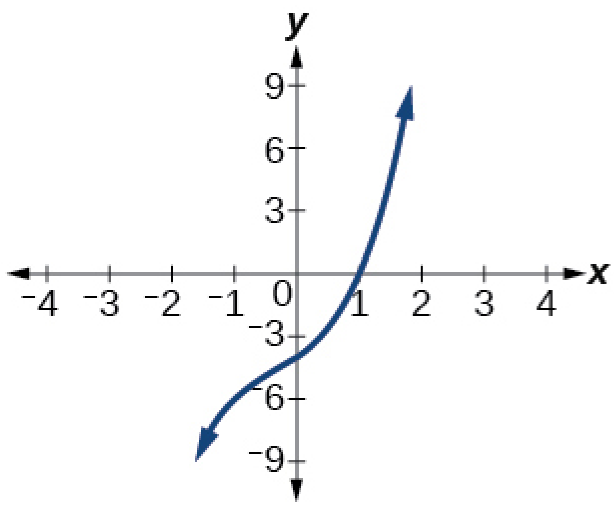
Exercise:

Problem: Factor is x^2-x+3



Exercise:

Problem: Factor is (x^2+2x+4)

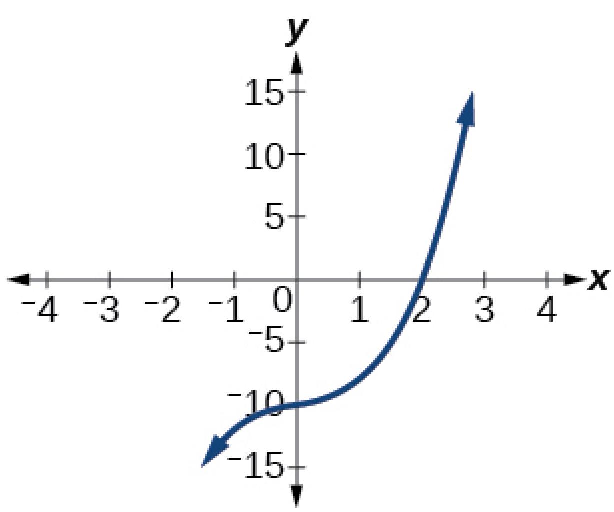


Solution:

$(x-1)(x^2+2x+4)$

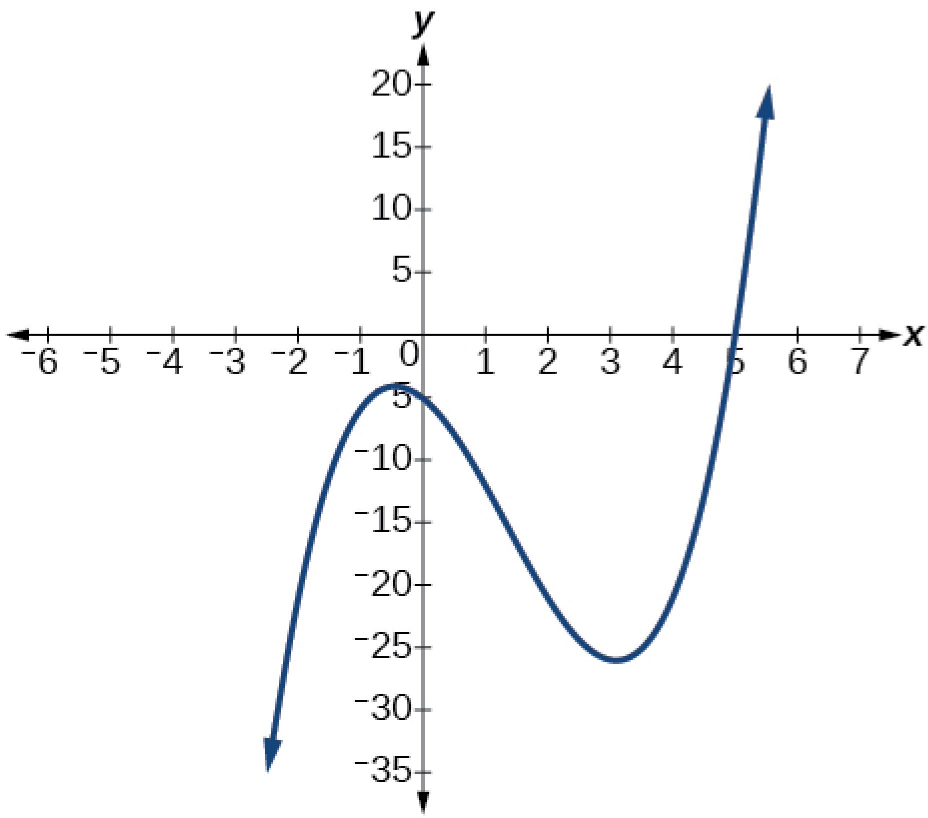
Exercise:

Problem: Factor is x^2+2x+5



Exercise:

Problem: Factor is $x^2 + x + 1$

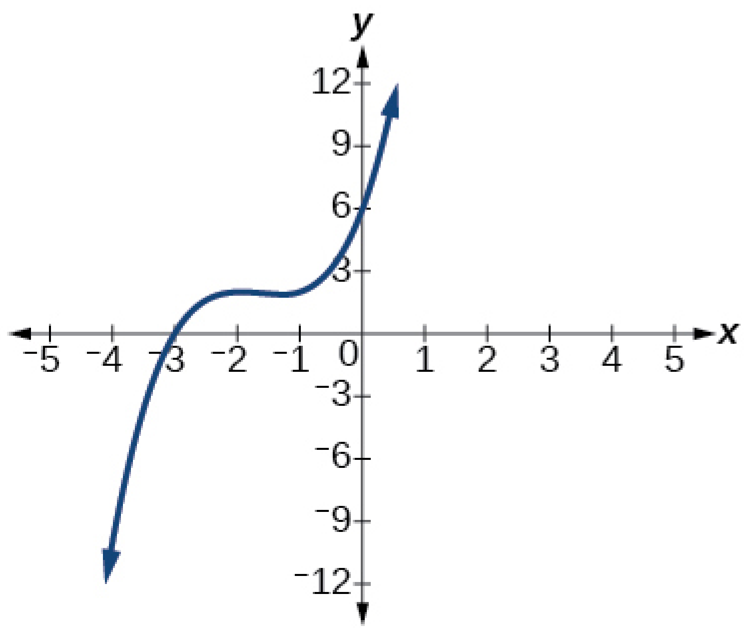


Solution:

$(x - 5)(x^2 + x + 1)$

Exercise:

Problem: Factor is$x^2 + 2x + 2$



For the following exercises, use synthetic division to find the quotient and remainder.

Exercise:

Problem: $\frac{4x^3-33}{x-2}$

Solution:

$\text{Quotient: } 4x^2+8x+16, \text{remainder: } -1$

Exercise:

Problem: $\frac{2x^3+25}{x+3}$

Exercise:

Problem: $\frac{3x^3+2x-5}{x-1}$

Solution:

$\text{Quotient: } 3x^2+3x+5, \text{remainder: } 0$

Exercise:

Problem: $\frac{-4x^3-x^2-12}{x+4}$

Exercise:

Problem: $\frac{x^4-22}{x+2}$

Solution:

$\text{Quotient:}\, x^3-2x^2+4x-8, \text{remainder:} \, -6$

Technology

For the following exercises, use a calculator with CAS to answer the questions.

Exercise:

Problem:

Consider $\frac{x^k-1}{x-1}$ with $k=1,\ 2,\ 3$. What do you expect the result to be if $k=4$?

Exercise:

Problem:

Consider $\frac{x^k+1}{x+1}$ for $k=1,\ 3,\ 5$. What do you expect the result to be if $k=7$?

Solution:

$x^6-x^5+x^4-x^3+x^2-x+1$

Exercise:

Problem:

Consider $\frac{x^4-k^4}{x-k}$ for $k=1,\ 2,\ 3.$ What do you expect the result to be if $k=4?$

Exercise:

Problem:

Consider $\frac{x^k}{x+1}$ with $k=1,\ 2,\ 3.$ What do you expect the result to be if $k=4?$

Solution:

$x^3-x^2+x-1+\frac{1}{x+1}$

Exercise:

Problem:

Consider $\frac{x^k}{x-1}$ with $k=1,\ 2,\ 3.$ What do you expect the result to be if $k=4?$

Extensions

For the following exercises, use synthetic division to determine the quotient involving a complex number.

Exercise:

Problem: $\frac{x+1}{x-i}$

Solution:

$1+\frac{1+i}{x-i}$

Exercise:

Problem: $\frac{x^2+1}{x-i}$

Exercise:

Problem: $\frac{x+1}{x+i}$

Solution:

$1 + \frac{1-i}{x+i}$

Exercise:

Problem: $\frac{x^2+1}{x+i}$

Exercise:

Problem: $\frac{x^3+1}{x-i}$

Solution:

$x^2 - ix - 1 + \frac{1-i}{x-i}$

Real-World Applications

For the following exercises, use the given length and area of a rectangle to express the width algebraically.

Exercise:

Problem: Length is $x + 5,$ area is $2x^2 + 9x - 5.$

Exercise:

Problem: Length is $2x \ + \ 5,$ area is $4x^3 + 10x^2 + 6x + 15$

Solution:

$2x^2 + 3$

Exercise:

Problem: Length is $3x–4$, area is $6x^4 - 8x^3 + 9x^2 - 9x - 4$

For the following exercises, use the given volume of a box and its length and width to express the height of the box algebraically.

Exercise:

Problem:

Volume is $12x^3 + 20x^2 - 21x - 36$, length is $2x + 3$, width is $3x - 4$.

Solution:

$2x + 3$

Exercise:

Problem:

Volume is $18x^3 - 21x^2 - 40x + 48$, length is $3x–4$, width is $3x–4$.

Exercise:

Problem:

Volume is $10x^3 + 27x^2 + 2x - 24$, length is $5x–4$, width is $2x + 3$.

Solution:

$x + 2$

Exercise:

Problem:

Volume is $10x^3 + 30x^2 - 8x - 24,$ length is $2,$ width is $x + 3.$

For the following exercises, use the given volume and radius of a cylinder to express the height of the cylinder algebraically.

Exercise:

Problem: Volume is $\pi(25x^3 - 65x^2 - 29x - 3),$ radius is $5x + 1.$

Solution:

$x - 3$

Exercise:

Problem: Volume is $\pi(4x^3 + 12x^2 - 15x - 50),$ radius is $2x + 5.$

Exercise:

Problem:

Volume is $\pi(3x^4 + 24x^3 + 46x^2 - 16x - 32),$ radius is $x + 4.$

Solution:

$3x^2 - 2$

Glossary

Division Algorithm
: given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x),$ there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = d(x)q(x) + r(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x).$

synthetic division

a shortcut method that can be used to divide a polynomial by a binomial of the form $x - k$

Graphs of Polynomial Functions
In this section, you will:

- Recognize characteristics of graphs of polynomial functions.
- Use factoring to find zeros of polynomial functions.
- Identify zeros and their multiplicities.
- Determine end behavior.
- Understand the relationship between degree and turning points.
- Graph polynomial functions.
- Use the Intermediate Value Theorem.

The revenue in millions of dollars for a fictional cable company from 2006 through 2013 is shown in [link].

Year	2006	2007	2008	2009	2010	2011	2012	2013
Revenues	52.4	52.8	51.2	49.5	48.6	48.6	48.7	47.1

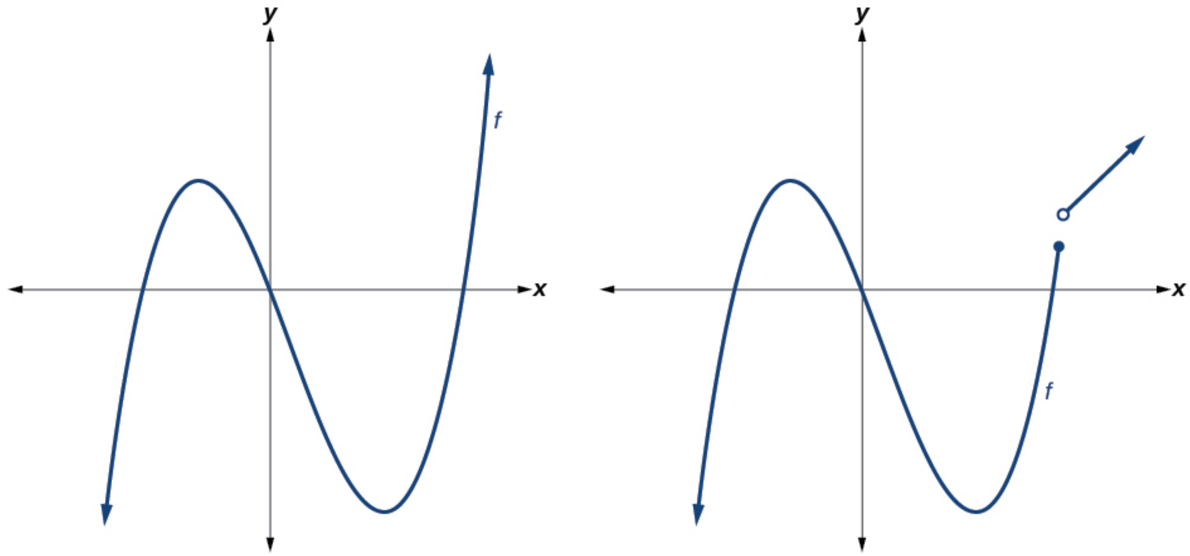
The revenue can be modeled by the polynomial function
Equation:

$$R(t) = -0.037t^4 + 1.414t^3 - 19.777t^2 + 118.696t - 205.332$$

where R represents the revenue in millions of dollars and t represents the year, with $t = 6$ corresponding to 2006. Over which intervals is the revenue for the company increasing? Over which intervals is the revenue for the company decreasing? These questions, along with many others, can be answered by examining the graph of the polynomial function. We have already explored the local behavior of quadratics, a special case of polynomials. In this section we will explore the local behavior of polynomials in general.

Recognizing Characteristics of Graphs of Polynomial Functions

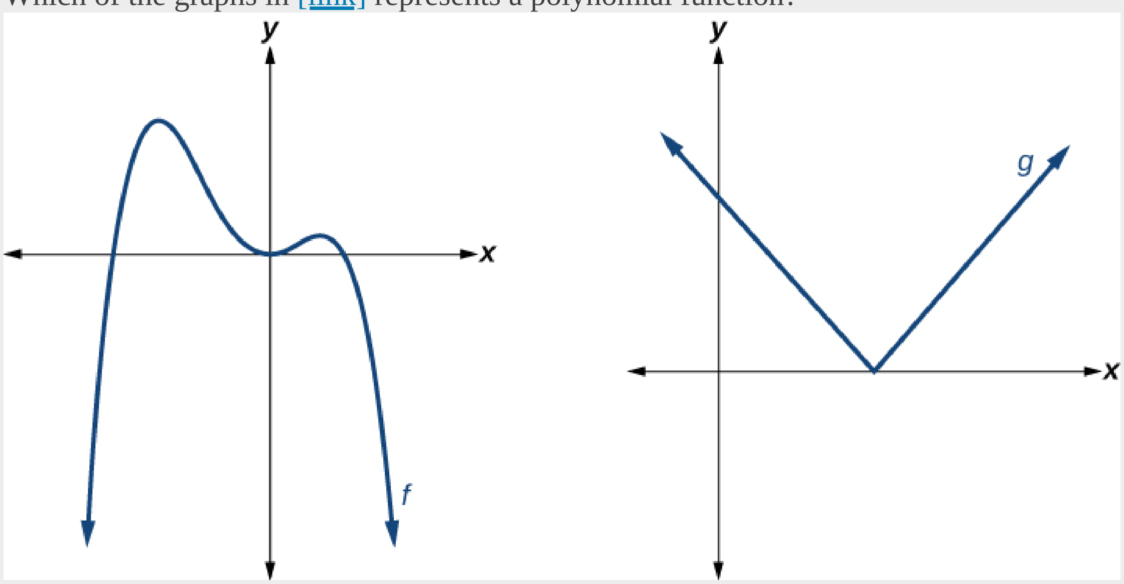
Polynomial functions of degree 2 or more have graphs that do not have sharp corners; recall that these types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous. [link] shows a graph that represents a polynomial function and a graph that represents a function that is not a polynomial.

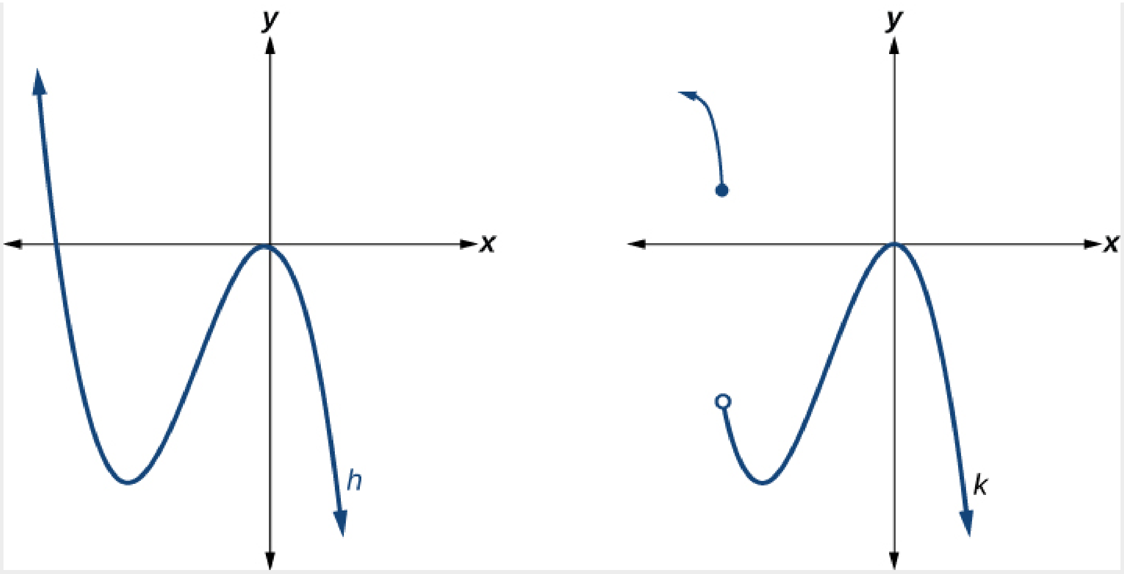


Example:
Exercise:

Problem:
Recognizing Polynomial Functions

Which of the graphs in [link] represents a polynomial function?





Solution:

The graphs of f and h are graphs of polynomial functions. They are smooth and continuous.

The graphs of g and k are graphs of functions that are not polynomials. The graph of function g has a sharp corner. The graph of function k is not continuous.

Note:
Do all polynomial functions have as their domain all real numbers?
Yes. Any real number is a valid input for a polynomial function.

Using Factoring to Find Zeros of Polynomial Functions

Recall that if f is a polynomial function, the values of x for which $f\left(x\right)=0$ are called zeros of $f.$ If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros**.**

We can use this method to find x-intercepts because at the x-intercepts we find the input values when the output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases in this section:

1. The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
2. The polynomial is given in factored form.

3. Technology is used to determine the intercepts.

Note:
Given a polynomial function $f,$ find the *x*-intercepts by factoring.

1. Set $f\left(x\right)=0.$
2. If the polynomial function is not given in factored form:
   a. Factor out any common monomial factors.
   b. Factor any factorable binomials or trinomials.
3. Set each factor equal to zero and solve to find the x-intercepts.

Example:
Exercise:

Problem:
Finding the *x*-Intercepts of a Polynomial Function by Factoring

Find the *x*-intercepts of $f(x)=x^6-3x^4+2x^2.$

Solution:

We can attempt to factor this polynomial to find solutions for $f\left(x\right)=0.$

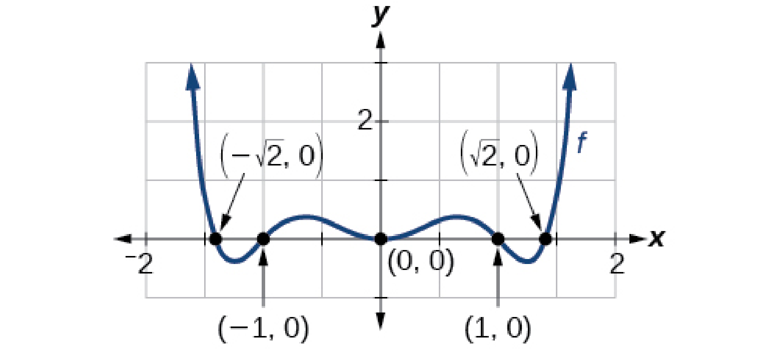
Equation:

$$\begin{array}{rl} x^6-3x^4+2x^2=0 & \text{Factor out the greatest common factor.} \\ x^2(x^4-3x^2+2)=0 & \text{Factor the trinomial.} \\ x^2(x^2-1)(x^2-2)=0 & \text{Set each factor equal to zero.} \end{array}$$

Equation:

$$\begin{array}{ccccc} & & (x^2-1)=0 & & (x^2-2)=0 \\ x^2=0 & \text{or} & x^2=1 & \text{or} & x^2=2 \\ x=0 & & x=\pm 1 & & x=\pm\sqrt{2} \end{array}$$

This gives us five x-intercepts: $(0,0), (1,0), (-1,0), (\sqrt{2},0),$ and $(-\sqrt{2},0)$. See [link]. We can see that this is an even function.



Example:
Exercise:

Problem:
Finding the *x*-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x) = x^3 - 5x^2 - x + 5$.

Solution:

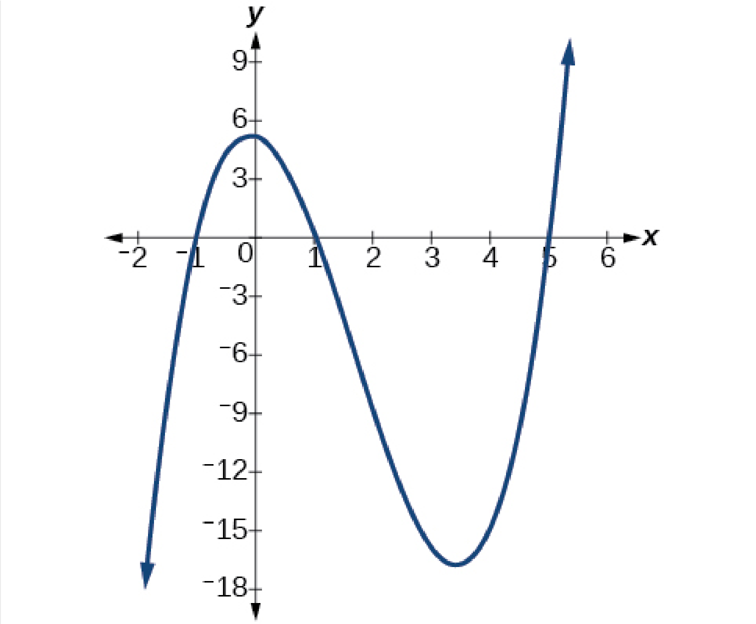
Find solutions for $f(x) = 0$ by factoring.
Equation:

$$\begin{array}{rl} x^3 - 5x^2 - x + 5 = 0 & \text{Factor by grouping.} \\ x^2(x-5) - (x-5) = 0 & \text{Factor out the common factor.} \\ (x^2 - 1)(x-5) = 0 & \text{Factor the difference of squares.} \\ (x+1)(x-1)(x-5) = 0 & \text{Set each factor equal to zero.} \end{array}$$

Equation:

$$\begin{array}{ccccc} x+1=0 & \text{or} & x-1=0 & \text{or} & x-5=0 \\ x=-1 & & x=1 & & x=5 \end{array}$$

There are three x-intercepts: $(-1,0), (1,0),$ and $(5,0)$. See [link].



Example:
Exercise:

Problem:
Finding the *y*- and *x*-Intercepts of a Polynomial in Factored Form

Find the y- and x-intercepts of $g(x) = (x - 2)^2(2x + 3)$.

Solution:

The y-intercept can be found by evaluating $g(0)$.
Equation:

$$\begin{aligned} g(0) &= (0 - 2)^2(2(0) + 3) \\ &= 12 \end{aligned}$$

So the y-intercept is $(0, 12)$.

The x-intercepts can be found by solving $g(x) = 0$.
Equation:

$$(x - 2)^2(2x + 3) = 0$$

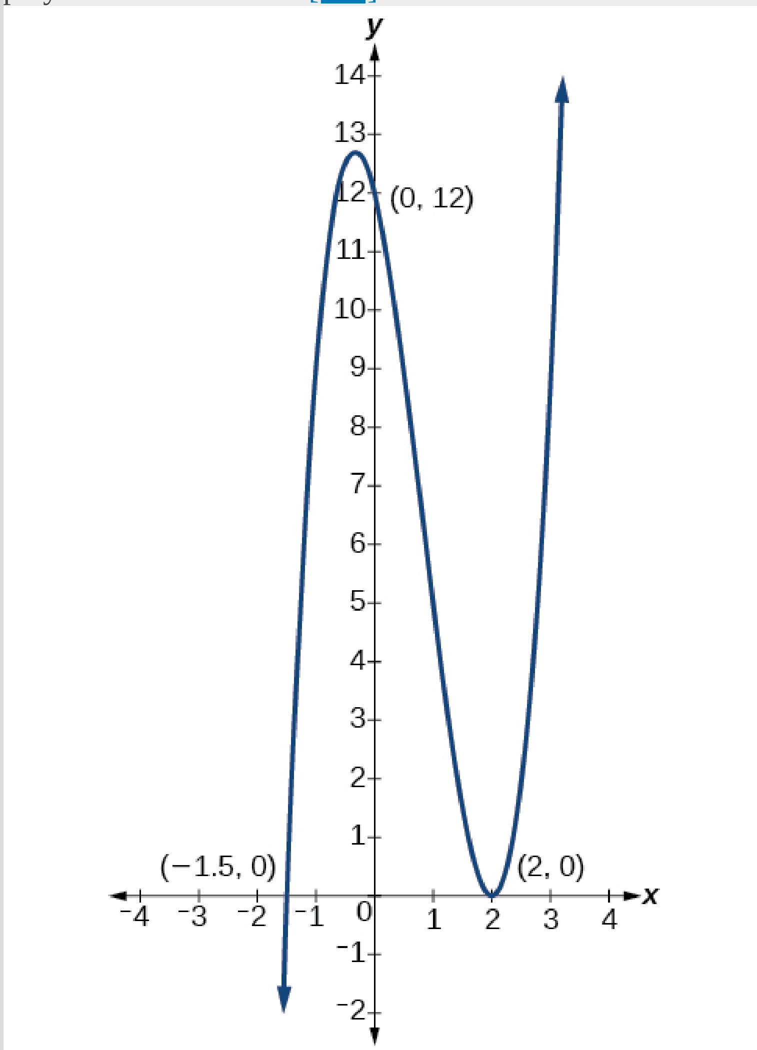
Equation:

$$
\begin{array}{rcr}
(x-2)^2 = 0 & & (2x+3) = 0 \\
x-2 = 0 & \text{or} & x = -\frac{3}{2} \\
x = 2 & &
\end{array}
$$

So the x-intercepts are $(2,0)$ and $\left(-\frac{3}{2},0\right)$.

Analysis

We can always check that our answers are reasonable by using a graphing calculator to graph the polynomial as shown in [link].



Example:
Exercise:

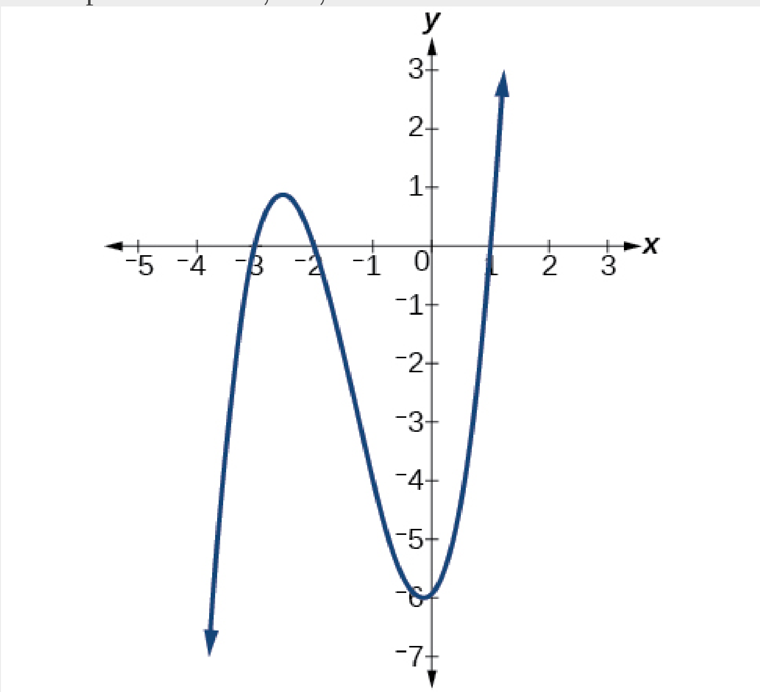
Problem:
Finding the *x*-Intercepts of a Polynomial Function Using a Graph

Find the x-intercepts of $h(x) = x^3 + 4x^2 + x - 6$.

Solution:

This polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques previously discussed. Fortunately, we can use technology to find the intercepts. Keep in mind that some values make graphing difficult by hand. In these cases, we can take advantage of graphing utilities.

Looking at the graph of this function, as shown in [link], it appears that there are *x*-intercepts at $x = -3, -2,$ and 1.



We can check whether these are correct by substituting these values for x and verifying that
Equation:

$$h(-3) = h(-2) = h(1) = 0.$$

Since $h(x) = x^3 + 4x^2 + x - 6,$ we have:
Equation:

$$\begin{aligned} h(-3) &= (-3)^3 + 4(-3)^2 + (-3) - 6 = -27 + 36 - 3 - 6 = 0 \\ h(-2) &= (-2)^3 + 4(-2)^2 + (-2) - 6 = -8 + 16 - 2 - 6 = 0 \\ h(1) &= (1)^3 + 4(1)^2 + (1) - 6 = 1 + 4 + 1 - 6 = 0 \end{aligned}$$

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

Equation:

$$\begin{aligned} h(x) &= x^3 + 4x^2 + x - 6 \\ &= (x+3)(x+2)(x-1) \end{aligned}$$

Note:
Exercise:

Problem: Find the y- and x-intercepts of the function $f(x) = x^4 - 19x^2 + 30x$.

Solution:

y-intercept $(0, 0)$; x-intercepts $(0, 0), (-5, 0), (2, 0),$ and $(3, 0)$

Identifying Zeros and Their Multiplicities

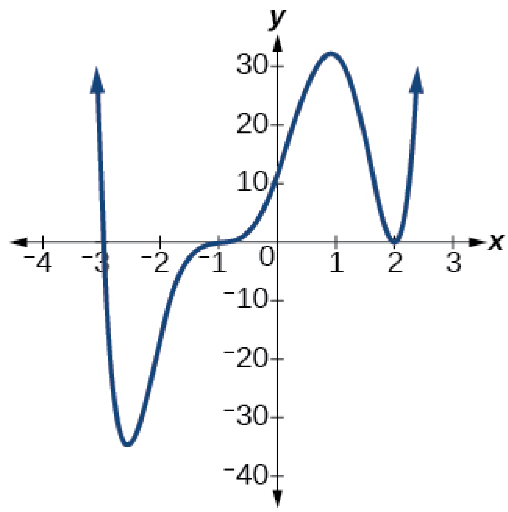
Graphs behave differently at various x-intercepts. Sometimes, the graph will cross over the horizontal axis at an intercept. Other times, the graph will touch the horizontal axis and bounce off.

Suppose, for example, we graph the function

Equation:

$$f(x) = (x+3)(x-2)^2(x+1)^3.$$

Notice in [link] that the behavior of the function at each of the x-intercepts is different.



Identifying the behavior of the graph at an *x*-intercept by examining the multiplicity of the zero.

The x-intercept -3 is the solution of equation $(x+3)=0$. The graph passes directly through the x-intercept at $x=-3$. The factor is linear (has a degree of 1), so the behavior near the intercept is like that of a line—it passes directly through the intercept. We call this a single zero because the zero corresponds to a single factor of the function.

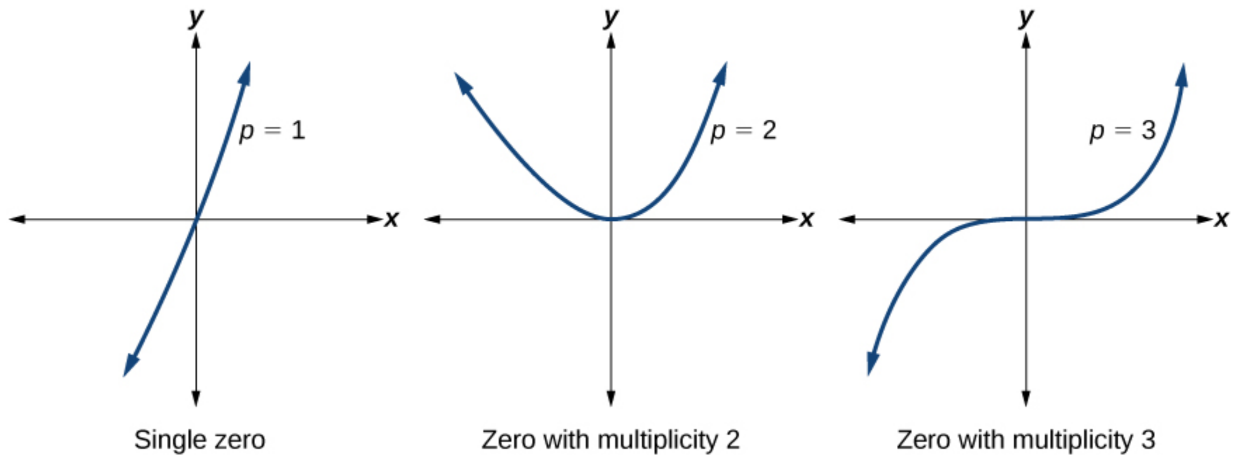
The x-intercept 2 is the repeated solution of equation $(x-2)^2=0$. The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic—it bounces off of the horizontal axis at the intercept.
Equation:

$$(x-2)^2=(x-2)(x-2)$$

The factor is repeated, that is, the factor $(x-2)$ appears twice. The number of times a given factor appears in the factored form of the equation of a polynomial is called the **multiplicity**. The zero associated with this factor, $x=2,$ has multiplicity 2 because the factor $(x-2)$ occurs twice.

The x-intercept -1 is the repeated solution of factor $(x+1)^3=0$. The graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic—with the same S-shape near the intercept as the toolkit function $f(x)=x^3.$ We call this a triple zero, or a zero with multiplicity 3.

For zeros with even multiplicities, the graphs *touch* or are tangent to the x-axis. For zeros with odd multiplicities, the graphs *cross* or intersect the x-axis. See [link] for examples of graphs of polynomial functions with multiplicity 1, 2, and 3.



For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches and leaves the x-axis.

For higher odd powers, such as 5, 7, and 9, the graph will still cross through the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis.

Note:
Graphical Behavior of Polynomials at x-Intercepts
If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the x-intercept h is determined by the power p. We say that $x=h$ is a zero of **multiplicity** p.
The graph of a polynomial function will touch the x-axis at zeros with even multiplicities. The graph will cross the x-axis at zeros with odd multiplicities.
The sum of the multiplicities is the degree of the polynomial function.

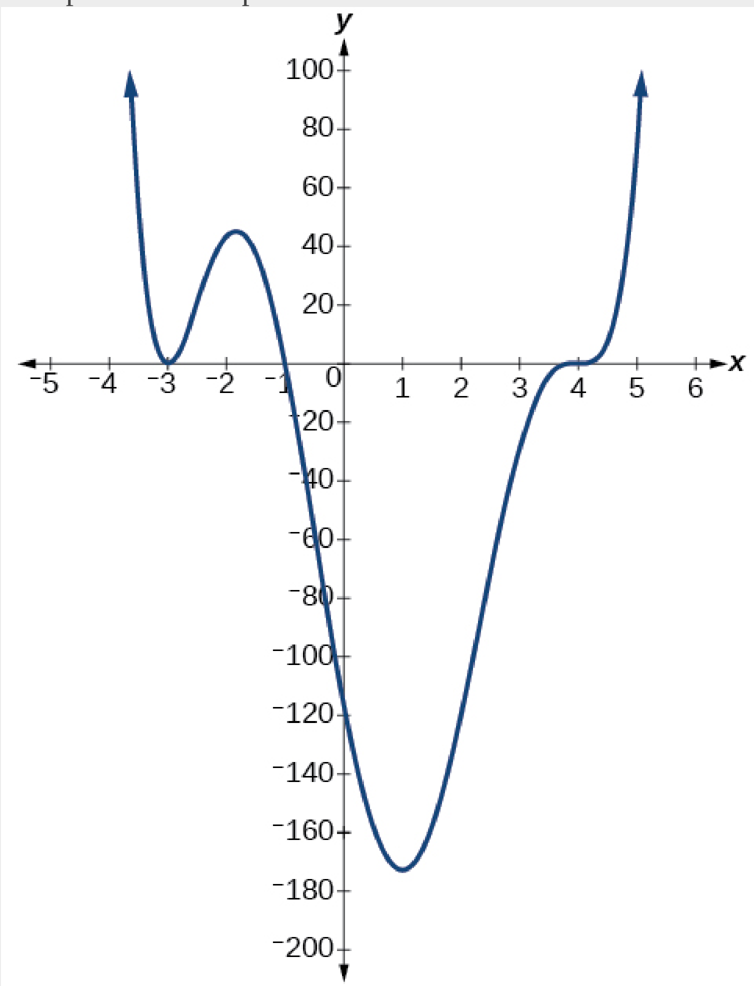
Note:
Given a graph of a polynomial function of degree n, identify the zeros and their multiplicities.

1. If the graph crosses the x-axis and appears almost linear at the intercept, it is a single zero.
2. If the graph touches the x-axis and bounces off of the axis, it is a zero with even multiplicity.
3. If the graph crosses the x-axis at a zero, it is a zero with odd multiplicity.
4. The sum of the multiplicities is n.

Example:
Exercise:

Problem:
Identifying Zeros and Their Multiplicities

Use the graph of the function of degree 6 in [link] to identify the zeros of the function and their possible multiplicities.



Solution:

The polynomial function is of degree n. The sum of the multiplicities must be n.

Starting from the left, the first zero occurs at $x = -3$. The graph touches the *x*-axis, so the multiplicity of the zero must be even. The zero of -3 has multiplicity 2.

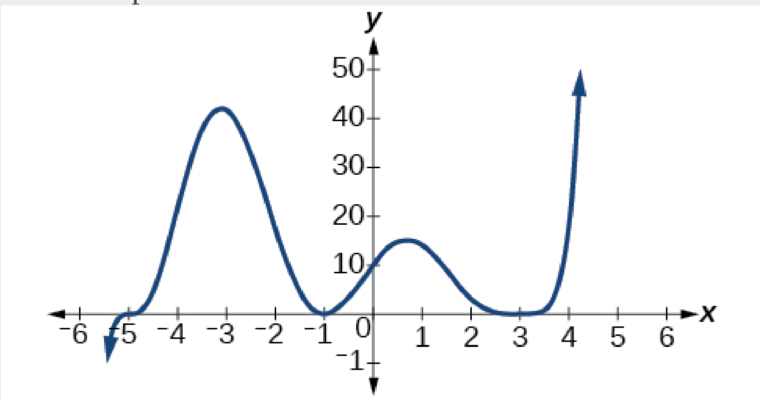
The next zero occurs at $x = -1$. The graph looks almost linear at this point. This is a single zero of multiplicity 1.

The last zero occurs at $x = 4$. The graph crosses the *x*-axis, so the multiplicity of the zero must be odd. We know that the multiplicity is likely 3 and that the sum of the multiplicities is likely 6.

Note:
Exercise:

Problem:

Use the graph of the function of degree 5 in [link] to identify the zeros of the function and their multiplicities.



Solution:

The graph has a zero of –5 with multiplicity 3, a zero of –1 with multiplicity 2, and a zero of 3 with multiplicity 4.

Determining End Behavior

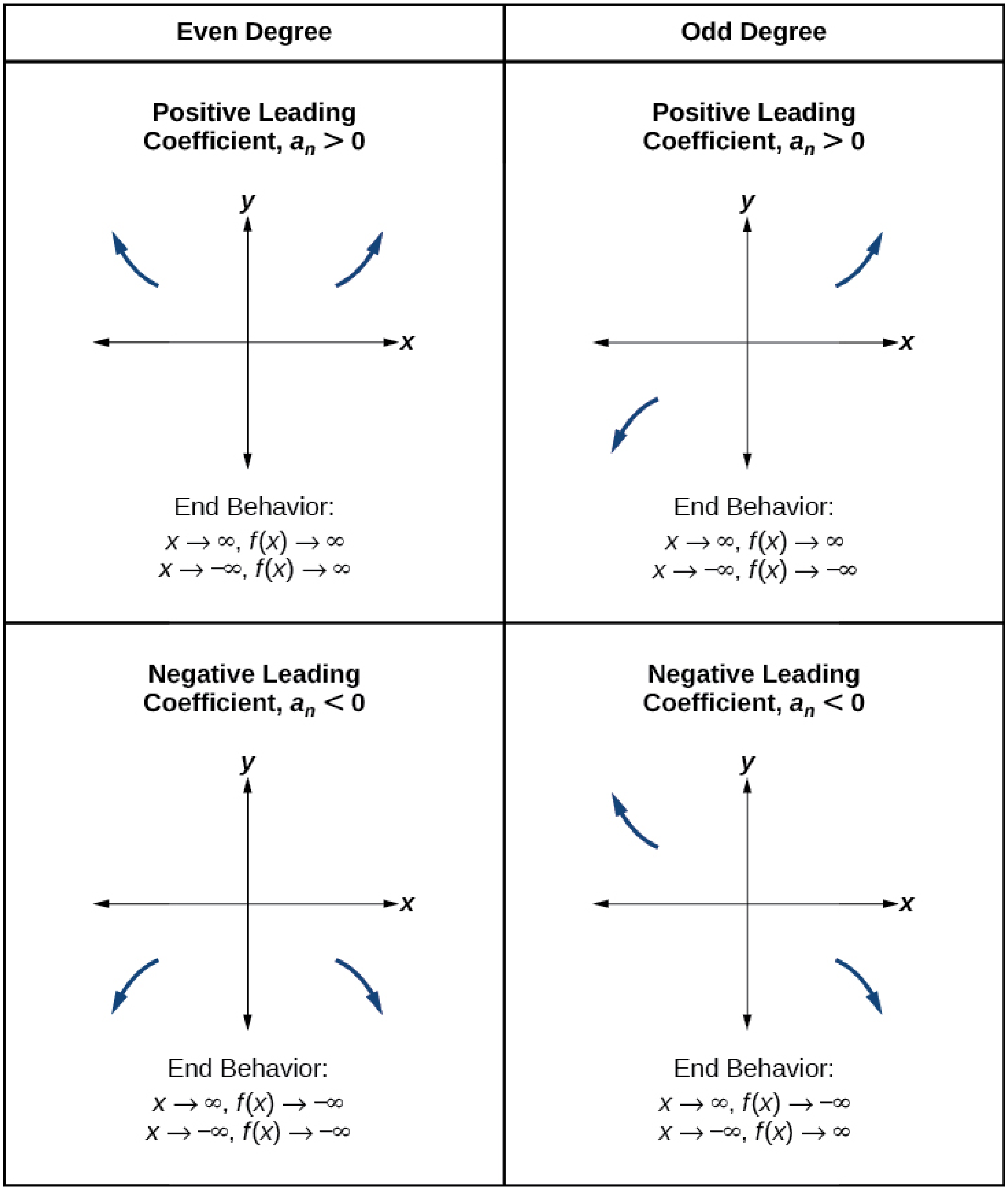
As we have already learned, the behavior of a graph of a polynomial function of the form

Equation:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$

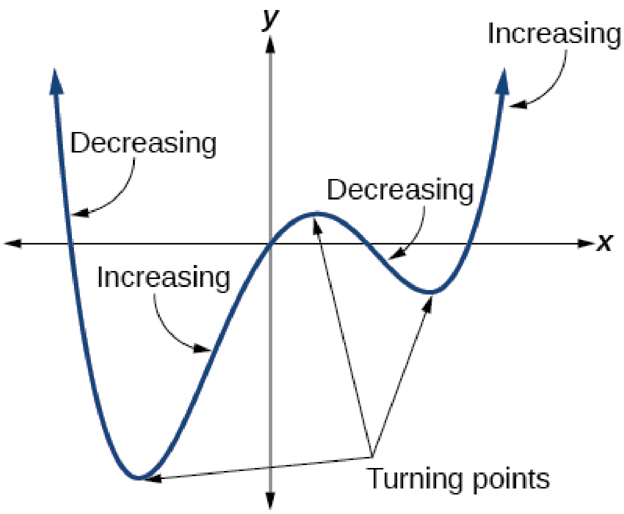
will either ultimately rise or fall as x increases without bound and will either rise or fall as x decreases without bound. This is because for very large inputs, say 100 or 1,000, the leading term dominates the size of the output. The same is true for very small inputs, say –100 or –1,000.

Recall that we call this behavior the *end behavior* of a function. As we pointed out when discussing quadratic equations, when the leading term of a polynomial function, $a_n x^n$, is an even power function, as x increases or decreases without bound, $f(x)$ increases without bound. When the leading term is an odd power function, as x decreases without bound, $f(x)$ also decreases without bound; as x increases without bound, $f(x)$ also increases without bound. If the leading term is negative, it will change the direction of the end behavior. [link] summarizes all four cases.



Understanding the Relationship between Degree and Turning Points

In addition to the end behavior, recall that we can analyze a polynomial function's local behavior. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function $f(x) = x^4 - x^3 - 4x^2 + 4x$ in [link]. The graph has three turning points.



This function f is a 4^{th} degree polynomial function and has 3 turning points. The maximum number of turning points of a polynomial function is always one less than the degree of the function.

Note:
Interpreting Turning Points
A turning point is a point of the graph where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising).
A polynomial of degree n will have at most $n-1$ turning points.

Example:
Exercise:

Problem:
Finding the Maximum Number of Turning Points Using the Degree of a Polynomial Function

Find the maximum number of turning points of each polynomial function.

a. $f(x)=-x^3+4x^5-3x^2+1$
b. $f(x)=-(x-1)^2\left(1+2x^2\right)$

Solution:

a. $f(x)=-x^{\ 3}+4x^5-3x^2+1$

First, rewrite the polynomial function in descending order:
$f(x)=4x^5-x^3-3x^2+1$

Identify the degree of the polynomial function. This polynomial function is of degree 5.

The maximum number of turning points is $5-1=4$.

b. $f(x)=-(x-1)^2\left(1+2x^2\right)$

First, identify the leading term of the polynomial function if the function were expanded.

$$f(x) = -(x-1)^2\ (1+2x^2)$$
$$a_n = -(x^2)\ (2x^2)\ -2x^4$$

Then, identify the degree of the polynomial function. This polynomial function is of degree 4.

The maximum number of turning points is $4-1=3$.

Graphing Polynomial Functions

We can use what we have learned about multiplicities, end behavior, and turning points to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

Note:
Given a polynomial function, sketch the graph.

1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetrical about the y-axis, that is, $f(-x)=f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x)=-f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.
4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch a graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

Example:
Exercise:

Problem:
Sketching the Graph of a Polynomial Function

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

Solution:

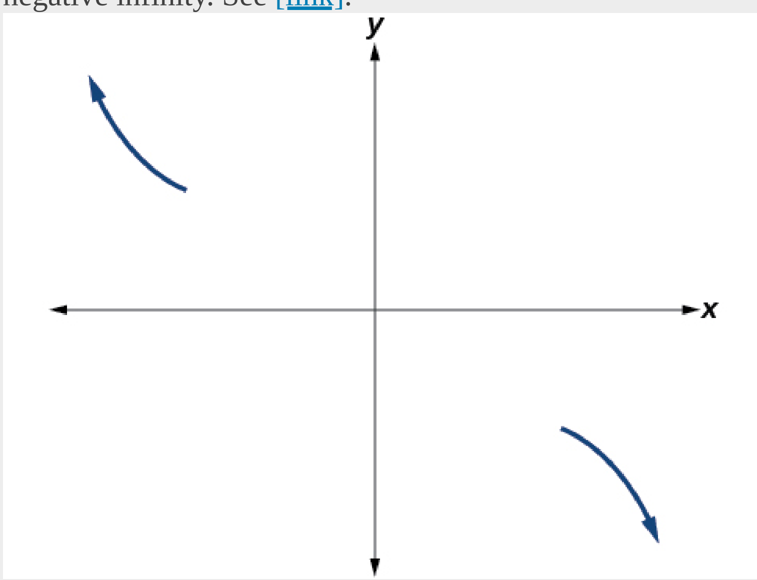
This graph has two x-intercepts. At $x = -3$, the factor is squared, indicating a multiplicity of 2. The graph will bounce at this x-intercept. At $x = 5$, the function has a multiplicity of one, indicating the graph will cross through the axis at this intercept.

The *y*-intercept is found by evaluating $f(0)$.
Equation:

$$\begin{aligned} f(0) &= -2(0+3)^2(0-5) \\ &= -2 \cdot 9 \cdot (-5) \\ &= 90 \end{aligned}$$

The y-intercept is $(0, 90)$.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x^3$, so the end behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs approach infinity, and the outputs increasing as the inputs approach negative infinity. See [link].

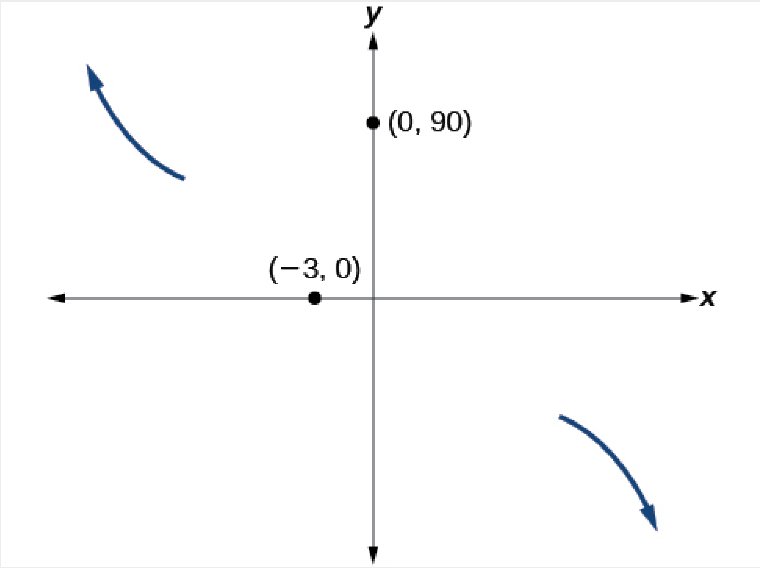


To sketch this, we consider that:

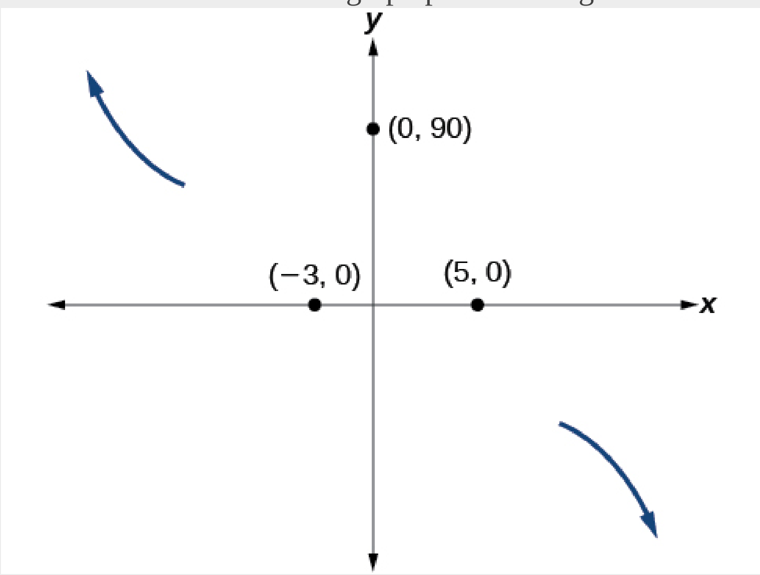
- As $x \to -\infty$ the function $f(x) \to \infty$, so we know the graph starts in the second quadrant and is decreasing toward the x-axis.

- Since $f(-x) = -2(-x+3)^2(-x-5)$ is not equal to $f(x)$, the graph does not display symmetry.
- At $(-3, 0)$, the graph bounces off of the x-axis, so the function must start increasing.

  At $(0, 90)$, the graph crosses the y-axis at the y-intercept. See [link].

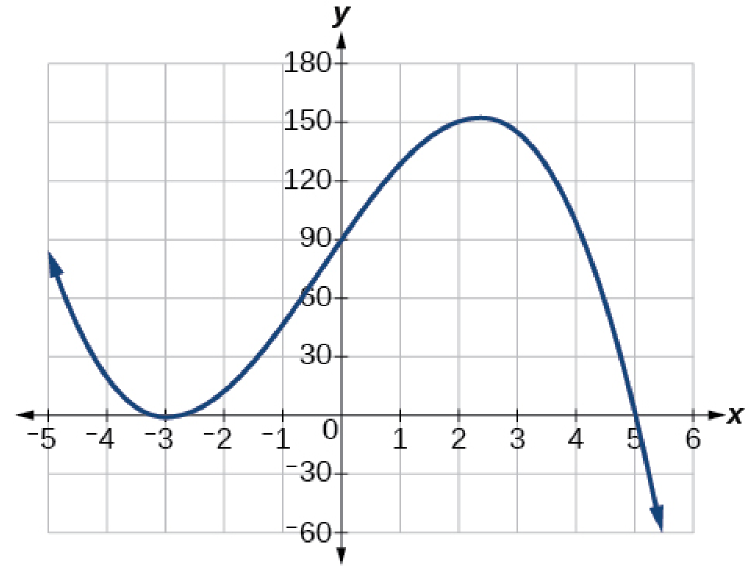


Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis because the graph passes through the next intercept at $(5, 0)$. See [link].



As $x \to \infty$ the function $f(x) \to -\infty$, so we know the graph continues to decrease, and we can stop drawing the graph in the fourth quadrant.

Using technology, we can create the graph for the polynomial function, shown in [link], and verify that the resulting graph looks like our sketch in [link].

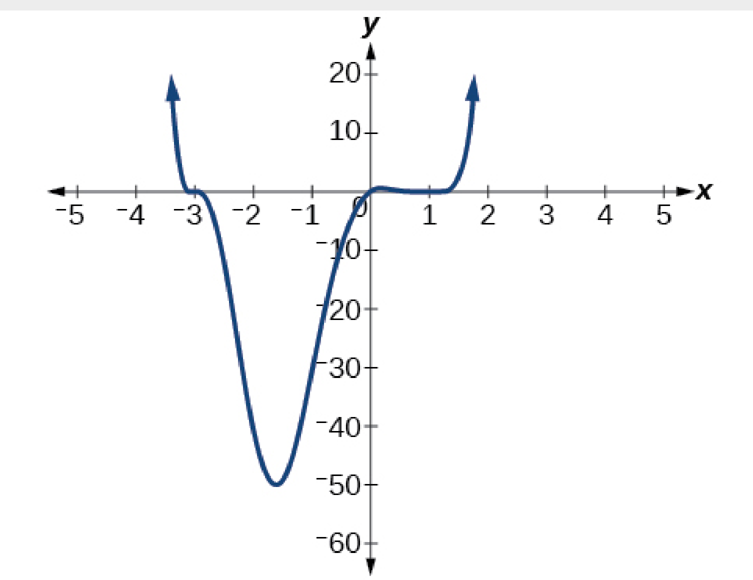


The complete graph of the polynomial function $f(x) = -2(x+3)^2(x-5)$

Note:
Exercise:

Problem: Sketch a graph of $f(x) = \frac{1}{4}x(x-1)^4(x+3)^3$.

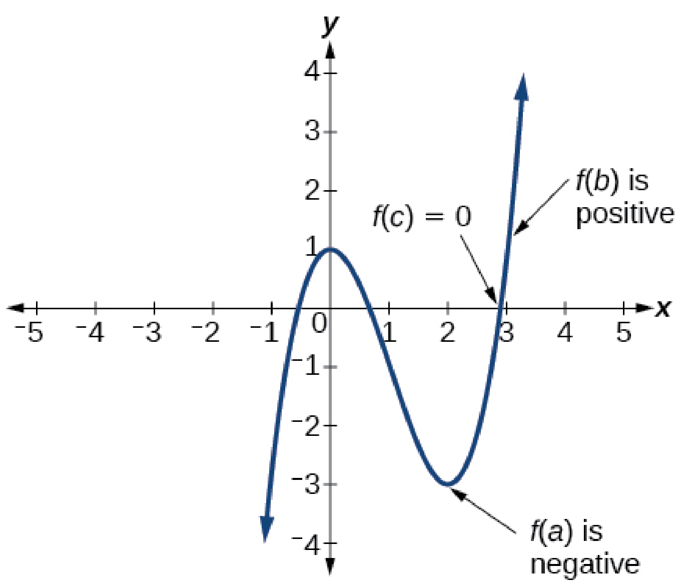
Solution:



Using the Intermediate Value Theorem

In some situations, we may know two points on a graph but not the zeros. If those two points are on opposite sides of the x-axis, we can confirm that there is a zero between them. Consider a polynomial function f whose graph is smooth and continuous. The **Intermediate Value Theorem** states that for two numbers a and b in the domain of f, if $a < b$ and $f\left(a\right) \neq f\left(b\right)$, then the function f takes on every value between $f\left(a\right)$ and $f\left(b\right)$. We can apply this theorem to a special case that is useful in graphing polynomial functions. If a point on the graph of a continuous function f at $x = a$ lies above the x-axis and another point at $x = b$ lies below the x-axis, there must exist a third point between $x = a$ and $x = b$ where the graph crosses the x-axis. Call this point $\left(c,\ f\left(c\right)\right)$. This means that we are assured there is a solution c where $f\left(c\right) = 0$.

In other words, the Intermediate Value Theorem tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis. [link] shows that there is a zero between a and b.



Using the Intermediate Value Theorem to show there exists a zero.

Note:
Intermediate Value Theorem

Let f be a polynomial function. The **Intermediate Value Theorem** states that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c) = 0$.

Example:
Exercise:

Problem:
Using the Intermediate Value Theorem

Show that the function $f(x) = x^3 - 5x^2 + 3x + 6$ has at least two real zeros between $x = 1$ and $x = 4$.

Solution:

As a start, evaluate $f(x)$ at the integer values $x = 1, 2, 3, \text{and} 4$. See [link].

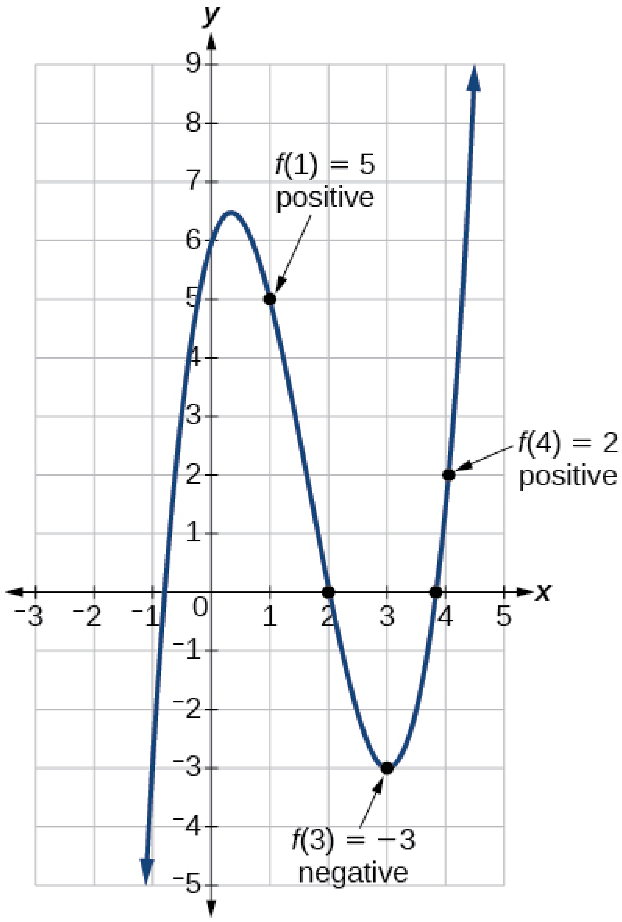
x	1	2	3	4
$f(x)$	5	0	–3	2

We see that one zero occurs at $x = 2$. Also, since $f(3)$ is negative and $f(4)$ is positive, by the Intermediate Value Theorem, there must be at least one real zero between 3 and 4.

We have shown that there are at least two real zeros between $x = 1$ and $x = 4$.

Analysis

We can also see on the graph of the function in [link] that there are two real zeros between $x = 1$ and $x = 4$.



Note:
Exercise:

Problem:

Show that the function $f(x) = 7x^5 - 9x^4 - x^2$ has at least one real zero between $x = 1$ and $x = 2$.

Solution:

Because f is a polynomial function and since $f(1)$ is negative and $f(2)$ is positive, there is at least one real zero between $x = 1$ and $x = 2$.

Writing Formulas for Polynomial Functions

Now that we know how to find zeros of polynomial functions, we can use them to write formulas based on graphs. Because a polynomial function written in factored form will have an x-intercept

where each factor is equal to zero, we can form a function that will pass through a set of x-intercepts by introducing a corresponding set of factors.

Note:
Factored Form of Polynomials
If a polynomial of lowest degree p has horizontal intercepts at $x = x_1, x_2, \ldots, x_n,$ then the polynomial can be written in the factored form: $f(x) = a(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}$ where the powers p_i on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor a can be determined given a value of the function other than the *x*-intercept.

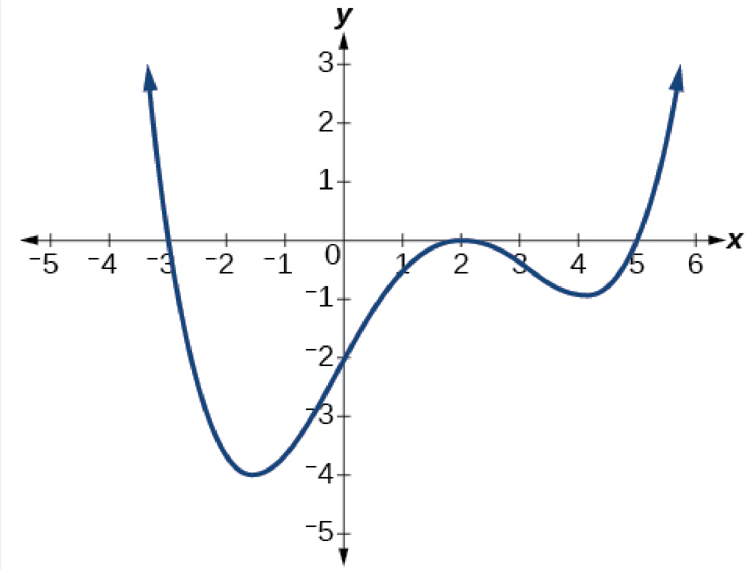
Note:
Given a graph of a polynomial function, write a formula for the function.

1. Identify the *x*-intercepts of the graph to find the factors of the polynomial.
2. Examine the behavior of the graph at the *x*-intercepts to determine the multiplicity of each factor.
3. Find the polynomial of least degree containing all the factors found in the previous step.
4. Use any other point on the graph (the *y*-intercept may be easiest) to determine the stretch factor.

Example:
Exercise:

Problem:
Writing a Formula for a Polynomial Function from the Graph

Write a formula for the polynomial function shown in [link].



Solution:

This graph has three x-intercepts: $x = -3, 2,$ and 5. The y-intercept is located at $(0, -2)$. At $x = -3$ and $x = 5,$ the graph passes through the axis linearly, suggesting the corresponding factors of the polynomial will be linear. At $x = 2,$ the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be second degree (quadratic). Together, this gives us

Equation:

$$f(x) = a(x+3)(x-2)^2(x-5)$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0, -2),$ to solve for a.

Equation:

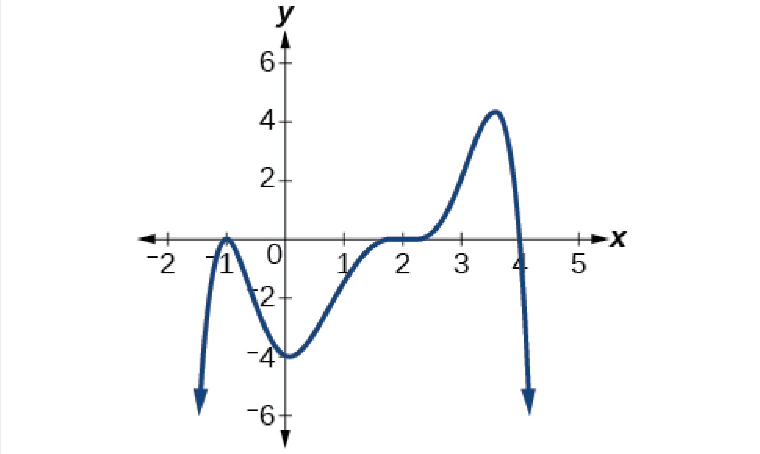
$$\begin{aligned} f(0) &= a(0+3)(0-2)^2(0-5) \\ -2 &= a(0+3)(0-2)^2(0-5) \\ -2 &= -60a \\ a &= \tfrac{1}{30} \end{aligned}$$

The graphed polynomial appears to represent the function $f(x) = \frac{1}{30}(x+3)(x-2)^2(x-5)$.

Note:

Exercise:

Problem: Given the graph shown in [link], write a formula for the function shown.



Solution:

$f(x) = -\frac{1}{8}(x-2)^3(x+1)^2(x-4)$

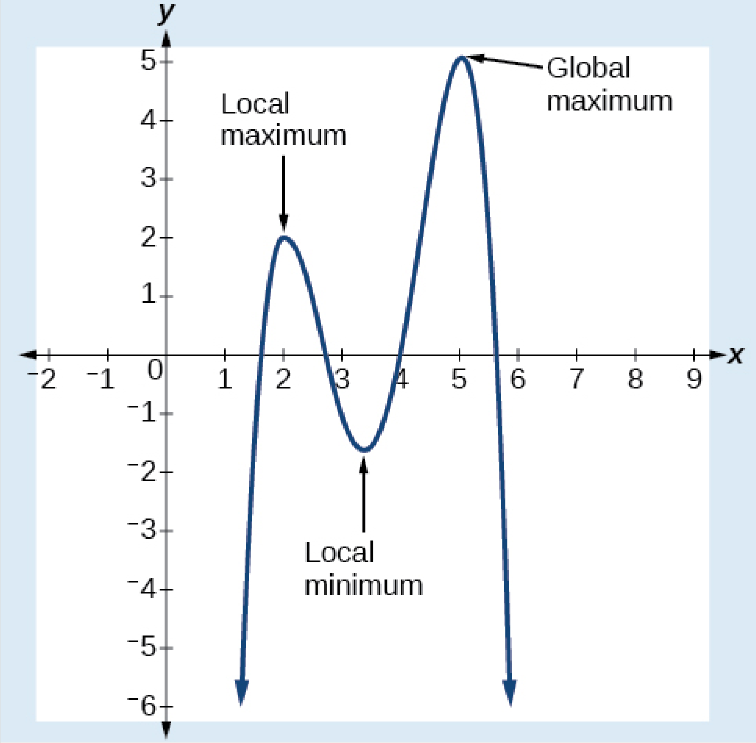
Using Local and Global Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Each turning point represents a local minimum or maximum. Sometimes, a turning point is the highest or lowest point on the entire graph. In these cases, we say that the turning point is a **global maximum** or a **global minimum**. These are also referred to as the absolute maximum and absolute minimum values of the function.

Note:
Local and Global Extrema
A local maximum or local minimum at $x = a$ (sometimes called the relative maximum or minimum, respectively) is the output at the highest or lowest point on the graph in an open interval around $x = a$. If a function has a local maximum at a, then $f(a) \geq f(x)$ for all x in an open interval around $x = a$. If a function has a local minimum at a, then $f(a) \leq f(x)$ for all x in an open interval around $x = a$.
A **global maximum** or **global minimum** is the output at the highest or lowest point of the function. If a function has a global maximum at a, then $f(a) \geq f(x)$ for all x. If a function has a global minimum at a, then $f(a) \leq f(x)$ for all x.

We can see the difference between local and global extrema in [link].



Note:
Do all polynomial functions have a global minimum or maximum?
No. Only polynomial functions of even degree have a global minimum or maximum. For example, $f(x) = x$ has neither a global maximum nor a global minimum.

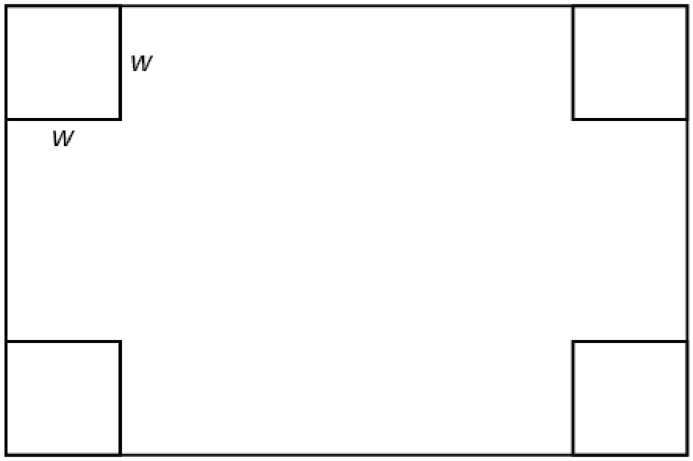
Example:
Exercise:

Problem:
Using Local Extrema to Solve Applications

An open-top box is to be constructed by cutting out squares from each corner of a 14 cm by 20 cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

Solution:

We will start this problem by drawing a picture like that in [link], labeling the width of the cut-out squares with a variable, w.

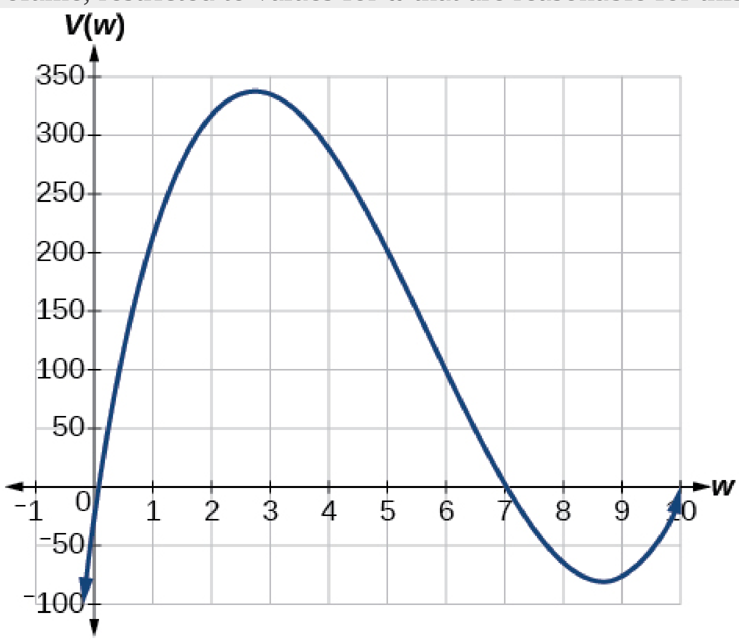


Notice that after a square is cut out from each end, it leaves a $(14 - 2w)$ cm by $(20 - 2w)$ cm rectangle for the base of the box, and the box will be w cm tall. This gives the volume

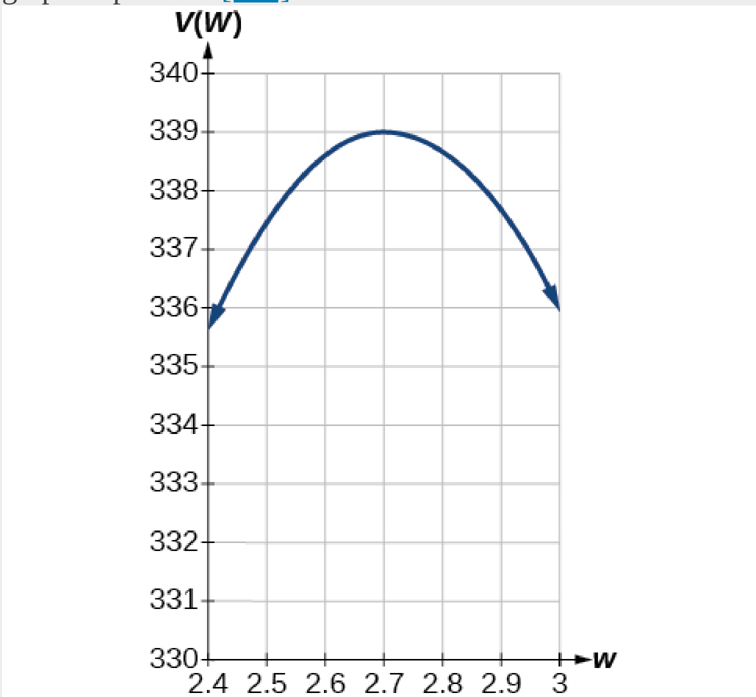
Equation:

$$\begin{aligned} V(w) &= (20 - 2w)(14 - 2w)w \\ &= 280w - 68w^2 + 4w^3 \end{aligned}$$

Notice, since the factors are $w,\ 20{-}2w$ and $14{-}2w,$ the three zeros are 10, 7, and 0, respectively. Because a height of 0 cm is not reasonable, we consider the only the zeros 10 and 7. The shortest side is 14 and we are cutting off two squares, so values w may take on are greater than zero or less than 7. This means we will restrict the domain of this function to $0 < w < 7$. Using technology to sketch the graph of $V\,(w)$ on this reasonable domain, we get a graph like that in [link]. We can use this graph to estimate the maximum value for the volume, restricted to values for w that are reasonable for this problem—values from 0 to 7.



From this graph, we turn our focus to only the portion on the reasonable domain, $[0,\ 7]$. We can estimate the maximum value to be around 340 cubic cm, which occurs when the squares are about 2.75 cm on each side. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph to produce [link].



From this zoomed-in view, we can refine our estimate for the maximum volume to about 339 cubic cm, when the squares measure approximately 2.7 cm on each side.

Note:
Exercise:

Problem:

Use technology to find the maximum and minimum values on the interval $[-1, 4]$ of the function $f(x) = -0.2(x-2)^3(x+1)^2(x-4)$.

Solution:

The minimum occurs at approximately the point $(0, -6.5)$, and the maximum occurs at approximately the point $(3.5, 7)$.

Note:

Access the following online resource for additional instruction and practice with graphing polynomial functions.

- Intermediate Value Theorem

Key Concepts

- Polynomial functions of degree 2 or more are smooth, continuous functions. See [link].
- To find the zeros of a polynomial function, if it can be factored, factor the function and set each factor equal to zero. See [link], [link], and [link].
- Another way to find the x-intercepts of a polynomial function is to graph the function and identify the points at which the graph crosses the x-axis. See [link].
- The multiplicity of a zero determines how the graph behaves at the x-intercepts. See [link].
- The graph of a polynomial will cross the horizontal axis at a zero with odd multiplicity.
- The graph of a polynomial will touch the horizontal axis at a zero with even multiplicity.
- The end behavior of a polynomial function depends on the leading term.
- The graph of a polynomial function changes direction at its turning points.
- A polynomial function of degree n has at most $n-1$ turning points. See [link].
- To graph polynomial functions, find the zeros and their multiplicities, determine the end behavior, and ensure that the final graph has at most $n-1$ turning points. See [link] and [link].
- Graphing a polynomial function helps to estimate local and global extremas. See [link].
- The Intermediate Value Theorem tells us that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c)=0$. See [link].

Section Exercises

Verbal

Exercise:

Problem:

What is the difference between an x-intercept and a zero of a polynomial function f?

Solution:

The x-intercept is where the graph of the function crosses the x-axis, and the zero of the function is the input value for which $f(x)=0$.

Exercise:

Problem:

If a polynomial function of degree n has n distinct zeros, what do you know about the graph of the function?

Exercise:

Problem:

Explain how the Intermediate Value Theorem can assist us in finding a zero of a function.

Solution:

If we evaluate the function at a and at b and the sign of the function value changes, then we know a zero exists between a and b.

Exercise:

Problem: Explain how the factored form of the polynomial helps us in graphing it.

Exercise:

Problem:

If the graph of a polynomial just touches the x-axis and then changes direction, what can we conclude about the factored form of the polynomial?

Solution:

There will be a factor raised to an even power.

Algebraic

For the following exercises, find the x- or t-intercepts of the polynomial functions.

Exercise:

Problem: $C\left(t\right)=2\left(t-4\right)\left(t+1\right)(t-6)$

Exercise:

Problem: $C\left(t\right)=3\left(t+2\right)\left(t-3\right)(t+5)$

Solution:

$(-2,0),(3,0),(-5,0)$

Exercise:

Problem: $C\left(t\right)=4t(t-2)^{2}(t+1)$

Exercise:

Problem: $C\left(t\right)=2t\left(t-3\right)(t+1)^{2}$

Solution:

$(3,0),(-1,0),(0,0)$

Exercise:

Problem: $C\left(t\right)=2t^4-8t^3+6t^2$

Exercise:

Problem: $C\left(t\right)=4t^4+12t^3-40t^2$

Solution:

$(0,0),\ \ (-5,0),\ \ (2,0)$

Exercise:

Problem: $f(x)=x^4-x^2$

Exercise:

Problem: $f(x)=x^3+x^2-20x$

Solution:

$(0,0),\ \ (-5,0),\ \ (4,0)$

Exercise:

Problem: $f(x)=x^3+6x^2-7x$

Exercise:

Problem: $f(x)=x^3+x^2-4x-4$

Solution:

$(2,0),\ \ (-2,0),\ \ (-1,0)$

Exercise:

Problem: $f(x)=x^3+2x^2-9x-18$

Exercise:

Problem: $f(x)=2x^3-x^2-8x+4$

Solution:

$(-2,0),\ (2,0),\ \left(\frac{1}{2},0\right)$

Exercise:

Problem: $f(x)=x^6-7x^3-8$

Exercise:

Problem: $f(x)=2x^4+6x^2-8$

Solution:

$(1,0),\ \ (-1,0)$

Exercise:

Problem: $f(x)=x^3-3x^2-x+3$

Exercise:

Problem: $f(x)=x^6-2x^4-3x^2$

Solution:

$(0,0),\ (\sqrt{3},0),\ (-\sqrt{3},0)$

Exercise:

Problem: $f(x)=x^6-3x^4-4x^2$

Exercise:

Problem: $f(x)=x^5-5x^3+4x$

Solution:

$(0,0),\ \ (1,0),\ (-1,0),\ \ (2,0),\ \ (-2,0)$

For the following exercises, use the Intermediate Value Theorem to confirm that the given polynomial has at least one zero within the given interval.

Exercise:

Problem: $f(x)=x^3-9x,$ between $x=-4$ and $x=-2.$

Exercise:

Problem: $f(x)=x^3-9x,$ between $x=2$ and $x=4.$

Solution:

$f(2) = -10$ and $f(4) = 28$. Sign change confirms.

Exercise:

Problem: $f(x) = x^5 - 2x,$ between $x = 1$ and $x = 2$.

Exercise:

Problem: $f(x) = -x^4 + 4,$ between $x = 1$ and $x = 3$.

Solution:

$f(1) = 3$ and $f(3) = -77.$ Sign change confirms.

Exercise:

Problem: $f(x) = -2x^3 - x,$ between $x = -1$ and $x = 1$.

Exercise:

Problem: $f(x) = x^3 - 100x + 2,$ between $x = 0.01$ and $x = 0.1$

Solution:

$f(0.01) = 1.000001$ and $f(0.1) = -7.999.$ Sign change confirms.

For the following exercises, find the zeros and give the multiplicity of each.

Exercise:

Problem: $f(x) = (x + 2)^3 (x - 3)^2$

Exercise:

Problem: $f(x) = x^2 (2x + 3)^5 (x - 4)^2$

Solution:

0 with multiplicity 2, $-\frac{3}{2}$ with multiplicity 5, 4 with multiplicity 2

Exercise:

Problem: $f(x) = x^3 (x - 1)^3 (x + 2)$

Exercise:

Problem: $f(x) = x^2 \left(x^2 + 4x + 4\right)$

Solution:

0 with multiplicity 2, –2 with multiplicity 2

Exercise:

Problem: $f(x) = (2x+1)^3 \left(9x^2 - 6x + 1\right)$

Exercise:

Problem: $f(x) = (3x+2)^5 \left(x^2 - 10x + 25\right)$

Solution:

$-\frac{2}{3}$ with multiplicity 5, 5 with multiplicity 2

Exercise:

Problem: $f(x) = x \left(4x^2 - 12x + 9\right) \left(x^2 + 8x + 16\right)$

Exercise:

Problem: $f(x) = x^6 - x^5 - 2x^4$

Solution:

0 with multiplicity 4, 2 with multiplicity 1, -1 with multiplicity 1

Exercise:

Problem: $f(x) = 3x^4 + 6x^3 + 3x^2$

Exercise:

Problem: $f(x) = 4x^5 - 12x^4 + 9x^3$

Solution:

$\frac{3}{2}$ with multiplicity 2, 0 with multiplicity 3

Exercise:

Problem: $f(x) = 2x^4 \left(x^3 - 4x^2 + 4x\right)$

Exercise:

Problem: $f(x)=4x^4\left(9x^4-12x^3+4x^2\right)$

Solution:

0 with multiplicity 6, $\frac{2}{3}$ with multiplicity 2

Graphical

For the following exercises, graph the polynomial functions. Note x- and y-intercepts, multiplicity, and end behavior.

Exercise:

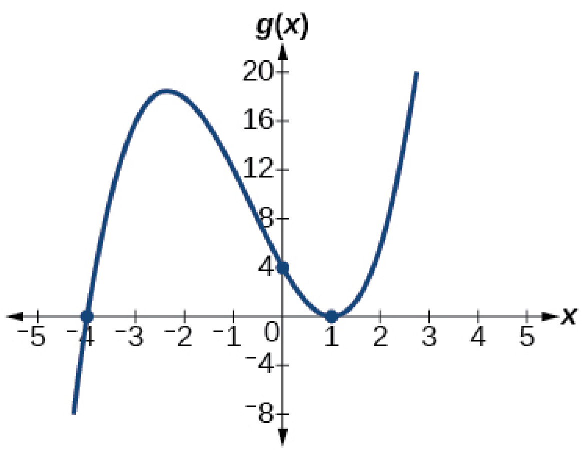
Problem: $f\left(x\right)={\left(x+3\right)}^{2}\left(x-2\right)$

Exercise:

Problem: $g\left(x\right)=\left(x+4\right){\left(x-1\right)}^{2}$

Solution:

x-intercepts, $\left(1,0\right)$ with multiplicity 2, $\left(-4,0\right)$ with multiplicity 1, y- intercept $\left(0,4\right)$. As $x\to -\infty$, $f\left(x\right)\to -\infty$, as $x\to \infty$, $f\left(x\right)\to \infty$.



Exercise:

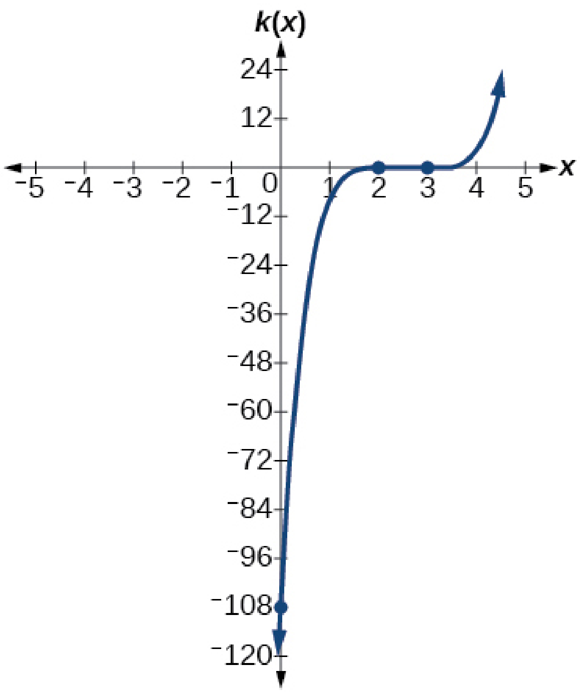
Problem: $h\left(x\right)={\left(x-1\right)}^{3}{\left(x+3\right)}^{2}$

Exercise:

Problem: $k\left(x\right)={\left(x-3\right)}^{3}{\left(x-2\right)}^{2}$

Solution:

x-intercepts $(3,0)$ with multiplicity 3, $(2,0)$ with multiplicity 2, y- intercept $(0,–108)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$.



Exercise:

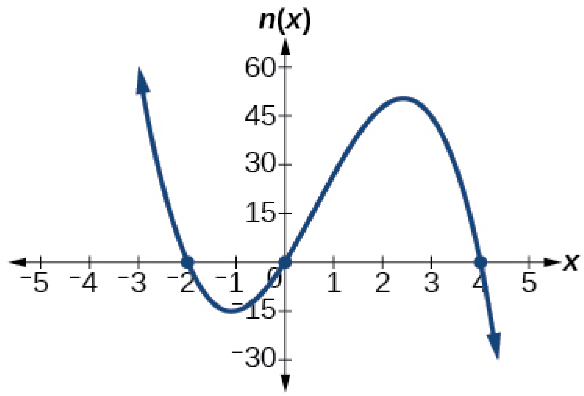
Problem:$m\left(x\right) = -2x\left(x - 1\right)(x + 3)$

Exercise:

Problem:$n\left(x\right) = -3x\left(x + 2\right)(x - 4)$

Solution:

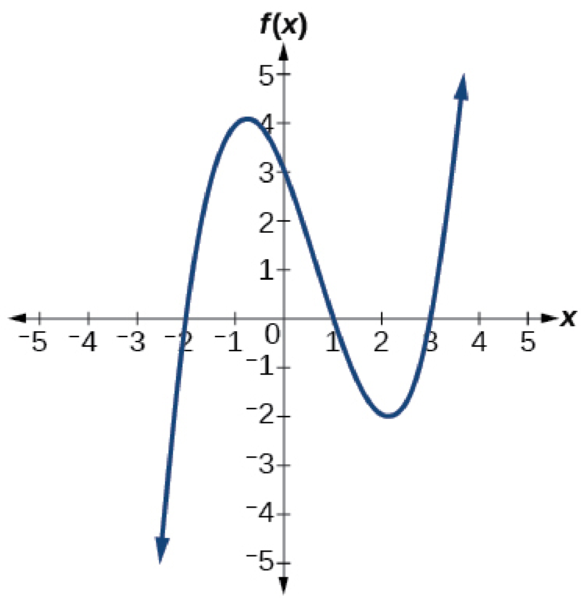
x-intercepts $(0,\,0),\;(-2,\,0),\;(4,\,0)$ with multiplicity 1, y-intercept $(0,\,0)$. As $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to -\infty$.



For the following exercises, use the graphs to write the formula for a polynomial function of least degree.

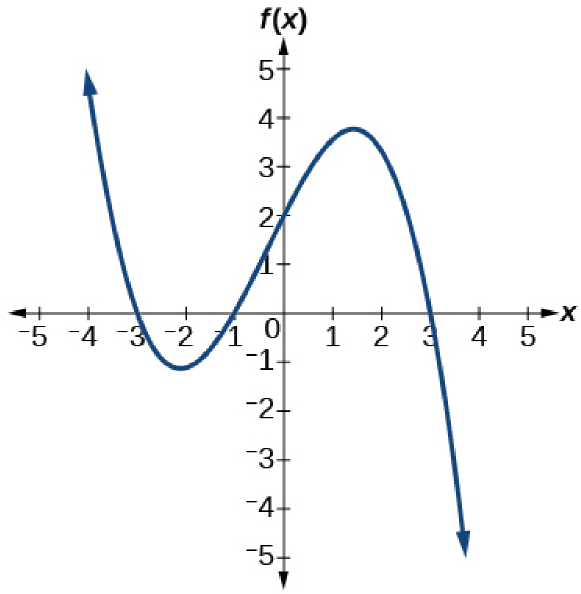
Exercise:

Problem:



Exercise:

Problem:

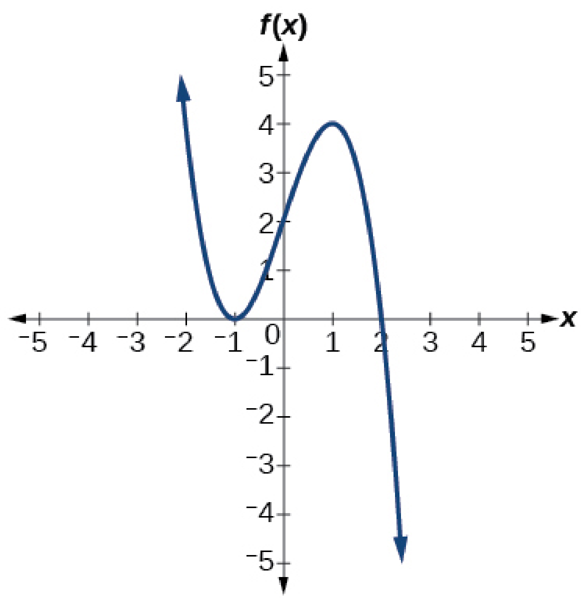


Solution:

$f(x) = -\frac{2}{9}(x-3)(x+1)(x+3)$

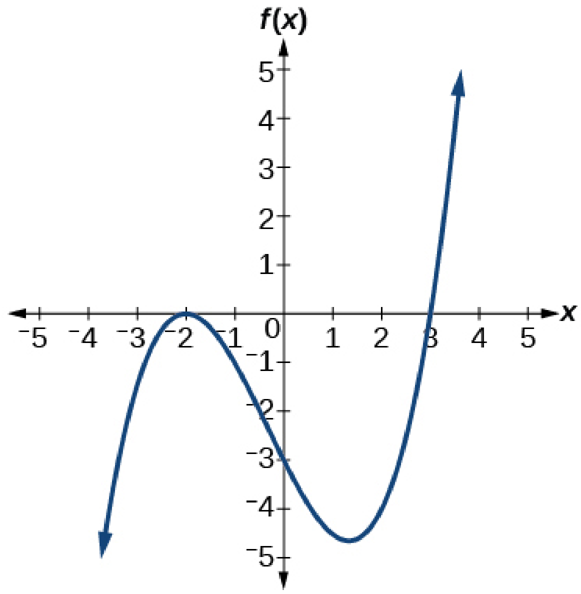
Exercise:

Problem:



Exercise:

Problem:

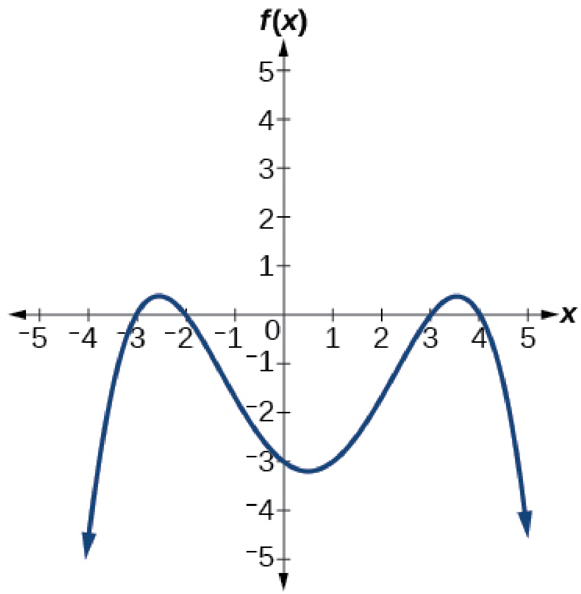


Solution:

$f(x) = \frac{1}{4}(x+2)^2(x-3)$

Exercise:

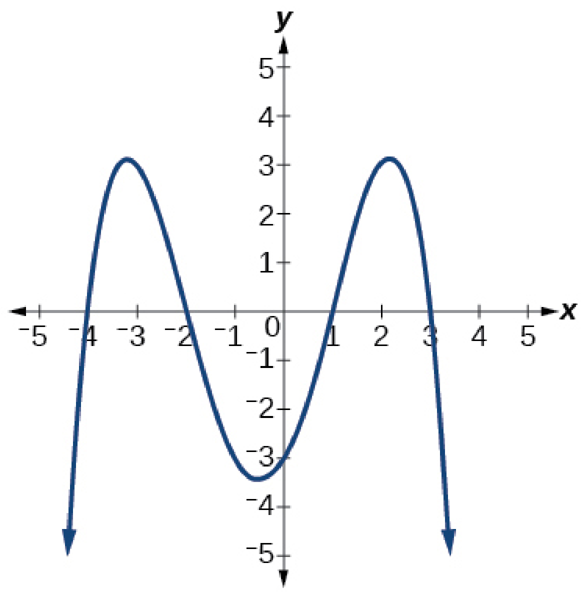
Problem:



For the following exercises, use the graph to identify zeros and multiplicity.

Exercise:

Problem:

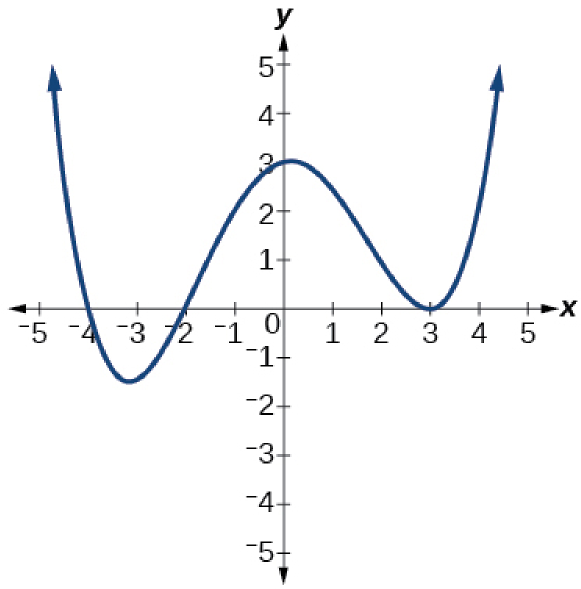


Solution:

–4, –2, 1, 3 with multiplicity 1

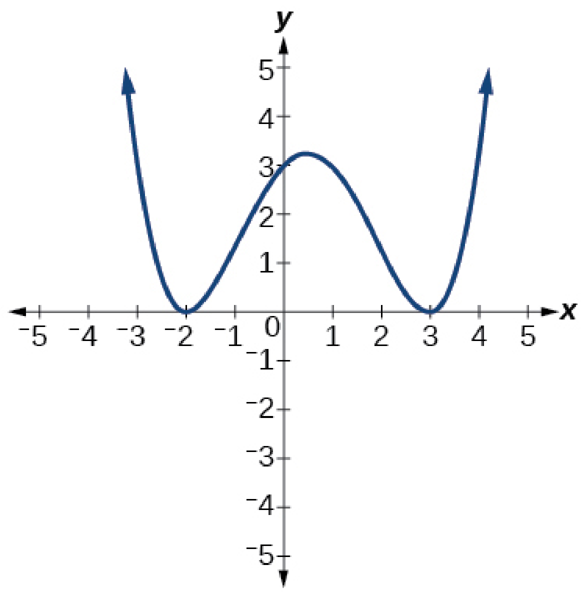
Exercise:

Problem:



Exercise:

Problem:

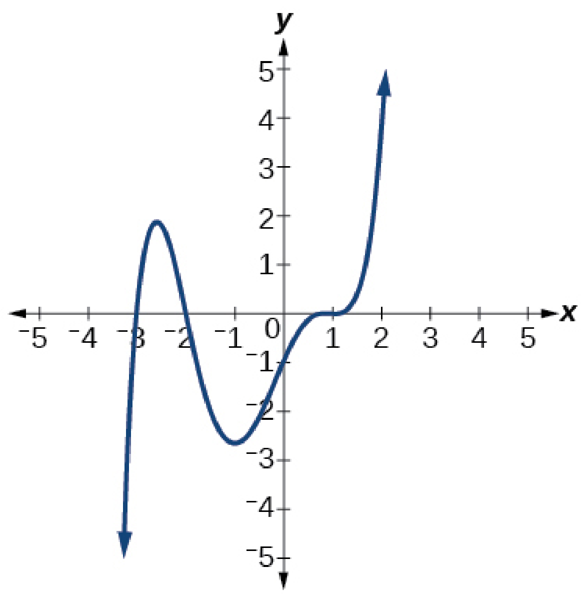


Solution:

–2, 3 each with multiplicity 2

Exercise:

Problem:



For the following exercises, use the given information about the polynomial graph to write the equation.

Exercise:

Problem: Degree 3. Zeros at $x=-2,\ x=1$, and $x=3$. y-intercept at $(0,-4)$.

Solution:

$f(x)=-\frac{2}{3}(x+2)(x-1)(x-3)$

Exercise:

Problem: Degree 3. Zeros at $x=-5,\ x=-2$,and $x=1$. y-intercept at $(0,6)$

Exercise:

Problem:

Degree 5. Roots of multiplicity 2 at $x=3$ and $x=1$, and a root of multiplicity 1 at $x=-3$. y-intercept at $(0,9)$

Solution:

$f(x)=\frac{1}{3}(x-3)^2(x-1)^2(x+3)$

Exercise:

Problem:

Degree 4. Root of multiplicity 2 at $x=4$, and a roots of multiplicity 1 at $x=1$ and $x=-2$. y-intercept at $(0,-3)$.

Exercise:

Problem:

Degree 5. Double zero at $x=1$, and triple zero at $x=3$. Passes through the point $(2,15)$.

Solution:

$f(x)=-15(x-1)^2(x-3)^3$

Exercise:

Problem: Degree 3. Zeros at $x=4$, $x=3$,and $x=2$. y-intercept at $(0,-24)$.

Exercise:

Problem: Degree 3. Zeros at $x=-3$, $x=-2$ and $x=1$. y-intercept at $(0,12)$.

Solution:

$f(x)=-2\,(x+3)\,(x+2)\,(x-1)$

Exercise:

Problem:

Degree 5. Roots of multiplicity 2 at $x=-3$ and $x=2$ and a root of multiplicity 1 at $x=-2$.

y-intercept at $(0,\ 4)$.

Exercise:

Problem:

Degree 4. Roots of multiplicity 2 at $x=\frac{1}{2}$ and roots of multiplicity 1 at $x=6$ and $x=-2$.

y-intercept at $(0{,}18)$.

Solution:

$f(x)=-\frac{3}{2}(2x-1)^2\,(x-6)\,(x+2)$

Exercise:

Problem: Double zero at $x=-3$ and triple zero at $x=0$. Passes through the point $(1,32)$.

Technology

For the following exercises, use a calculator to approximate local minima and maxima or the global minimum and maximum.

Exercise:

Problem: $f(x)=x^3-x-1$

Solution:

local max $(-.58,\,-.62)$, local min $(.58,\,-1.38)$

Exercise:

Problem: $f(x)=2x^3-3x-1$

Exercise:

Problem: $f(x)=x^4+x$

Solution:

global min $(-.63,\,-.47)$

Exercise:

Problem: $f(x) = -x^4 + 3x - 2$

Exercise:

Problem: $f(x) = x^4 - x^3 + 1$

Solution:

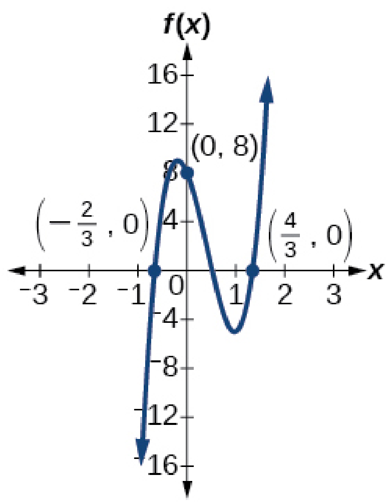
global min $(.75, .89)$

Extensions

For the following exercises, use the graphs to write a polynomial function of least degree.

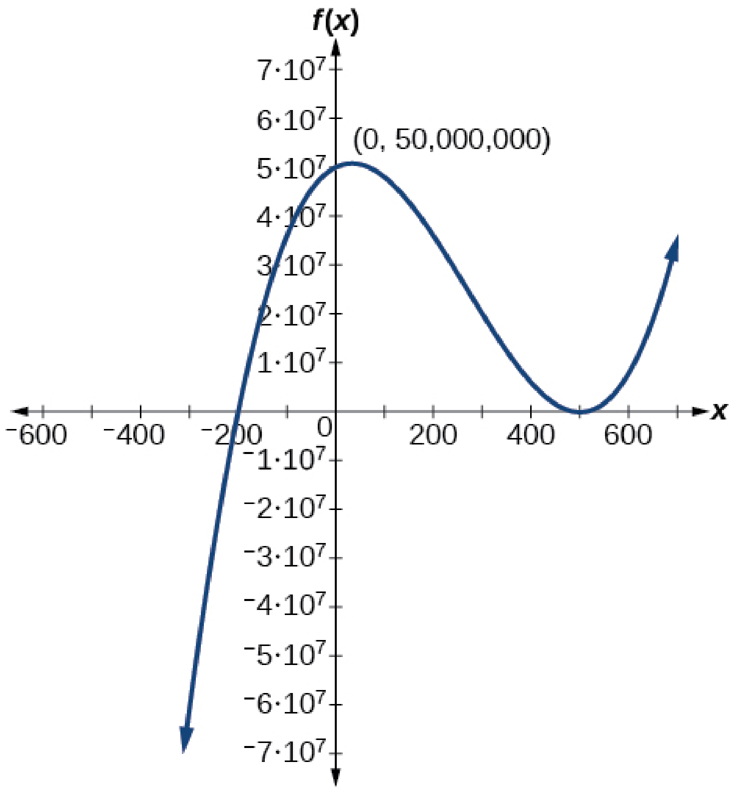
Exercise:

Problem:



Exercise:

Problem:

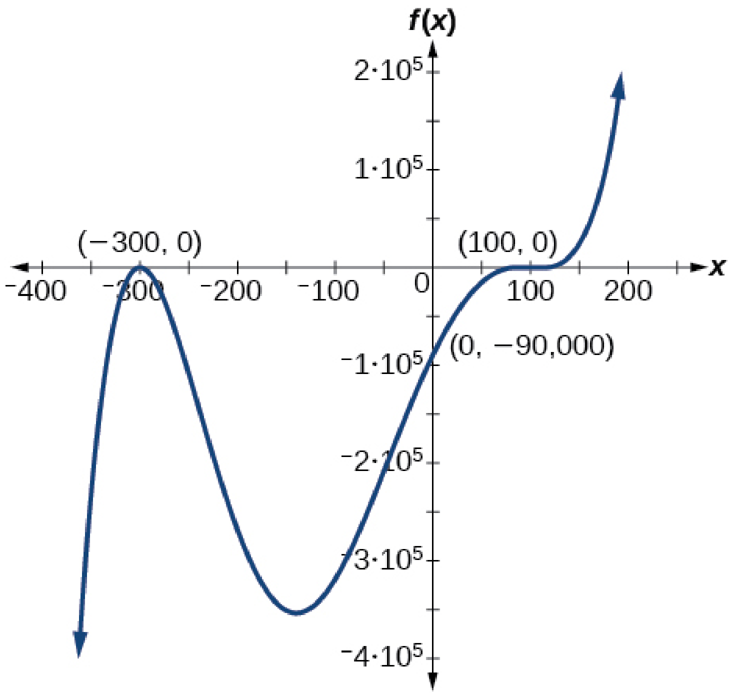


Solution:

$f(x) = (x - 500)^2 (x + 200)$

Exercise:

Problem:



Real-World Applications

For the following exercises, write the polynomial function that models the given situation.

Exercise:

Problem:

A rectangle has a length of 10 units and a width of 8 units. Squares of x by x units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a polynomial function in terms of x.

Solution:

$f(x) = 4x^3 - 36x^2 + 80x$

Exercise:

Problem:

Consider the same rectangle of the preceding problem. Squares of $2x$ by $2x$ units are cut out of each corner. Express the volume of the box as a polynomial in terms of x.

Exercise:

Problem:

A square has sides of 12 units. Squares $x+1$ by $x+1$ units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a function in terms of x.

Solution:

$f(x)=4x^3-36x^2+60x+100$

Exercise:

Problem:

A cylinder has a radius of $x+2$ units and a height of 3 units greater. Express the volume of the cylinder as a polynomial function.

Exercise:

Problem:

A right circular cone has a radius of $3x+6$ and a height 3 units less. Express the volume of the cone as a polynomial function. The volume of a cone is $V=\frac{1}{3}\pi r^2h$ for radius r and height h.

Solution:

$f(x)=\pi(9x^3+45x^2+72x+36)$

Glossary

global maximum
: highest turning point on a graph; $f(a)$ where $f(a)\geq f(x)$ for all x.

global minimum
: lowest turning point on a graph; $f(a)$ where $f(a)\leq f(x)$ for all x.

Intermediate Value Theorem
: for two numbers a and b in the domain of f, if $a<b$ and $f(a)\neq f(b)$, then the function f takes on every value between $f(a)$ and $f(b)$; specifically, when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis

multiplicity
: the number of times a given factor appears in the factored form of the equation of a polynomial; if a polynomial contains a factor of the form $(x-h)^p$, $x=h$ is a zero of multiplicity p.